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Sheaf Cohomology and the De Rham Theorem

BACHELOR THESIS

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Wiskunde

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathcal{C}^0 & \longrightarrow & \mathcal{C}^1 & \longrightarrow & \mathcal{C}^2 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^0 & \longrightarrow & J^{0,0} & \longrightarrow & J^{0,1} & \longrightarrow & J^{0,2} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^1 & \longrightarrow & J^{1,0} & \longrightarrow & J^{1,1} & \longrightarrow & J^{1,2} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^2 & \longrightarrow & J^{2,0} & \longrightarrow & J^{2,1} & \longrightarrow & J^{2,2} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \vdots & & \end{array}$$

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Abstract

Cohomology is a tool which is used across mathematics. Cohomology theories such as singular and De Rham cohomologies are invariant on topological spaces and manifolds respectively. Sheaf cohomology generalises these theories, which we introduce in this thesis. We begin by defining sheaves, which will be associated to a given topological space. As we will see, the sheaf axioms embody properties useful in solving local to global problems. In fact, we will introduce a class of sheaves for which the existence of partitions of unity is inherent. Using the theory of sheaves, we generalize cohomology theory to the sheaf cohomology, which resolves the cohomology of a space via a sheaf over said space. This will be carefully defined via homological algebra, a branch of mathematics concerned with the algebraic study of homology and cohomology. We then move on to exploring some of the possibilities of this theory, re-establishing some of the facts familiar to cohomology theory. We conclude the thesis with a proof of the De Rham theorem, which states that the singular and de Rham cohomology of a manifold are isomorphic; to sketch potential applications of this theory.

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1 Introduction

Invariants are an important tool in mathematics, as they are a useful in answering questions such as, whether or not a 3-sphere and torus are homeomorphic. We have seen many examples of such invariants in the bachelor, for example, in topology topological properties such as being Hausdorff or compact were invariants as was the number of connected components of a topological space. In this thesis we aim to introduce one another such invariant, which is called the sheaf cohomology. This invariant will assign a sequence of modules to a topological space X . When these modules are vector spaces, this in particular gives us numerical invariants such as the Euler characteristic.

In the first chapter, we introduce sheaf theory. A presheaf $\mathcal{F}$ over a topological space X is a mathematical objects which assign, sets, groups, rings, etc. to opens $U \subset X$, in a manner that is consistent with some form of restriction map. Then $\mathcal{F}$ is said to a sheaf if it satisfies two properties. These sheaf axioms encode useful results that can be used to extend local properties globally. Many results pertaining to sheaves will involve solving such problems. Sheaves will be defined by means of presheaves and we will describe several methods for obtaining a sheaf from a presheaf which each give their different interpretations of sheaves. We will also describe the sheaf category beginning with sheaves of sets and later extending the theory to sheaves of modules. Finally, we finish the chapter by providing some examples of sheaves one of which will be important in establishing the connection between the sheaf and de Rham cohomology.

In the second chapter, we will define the sheaf cohomology of a topological space X . From here on, we will work with sheaves of some algebraic structure, to ensure the existence of the kernel of a sheaf morphism. This is a necessary condition for defining exactness. We introduce some homological algebra concepts with proofs which will allow us to properly define sheaf cohomology. These proofs will require some additional theory which we discuss here. Using these tools the notion of cohomology being defined via an injective resolution will be defined and shown to be meaningful.

The third chapter is devoted to results in sheaf cohomology. We will discuss the functoriality of sheaf cohomology, and establish that this is an invariant up to homotopy for locally constant sheaves of abelian groups. Several other results will be discussed such as how we can obtain a long exact sequence in cohomology from a short exact sequence of sheaves, this result will be important in proving some of our later results. We then end the chapter by defining two new classes of acyclic sheaves. Of importance here is that we show that resolving a sheaf by any class of acyclic sheaf induces a cohomology isomorphic to the sheaf cohomology. The theory of this chapter is developed in greater generality than required for the final chapter.

In the last chapter, we apply the developed theory to proving the de Rham theorem, which posits that the singular and de Rham cohomologies of any manifold are isomorphic. Our aim is that this motivates the theory of sheaf cohomology developed. Also, we define singular homology and establish some basic facts.

We have striven to assume minimal prior knowledge of mathematical concepts beyond the second year bachelor level. Unfortunately, this was not possible at all points in the thesis. In particular, we assume the reader has a basic working knowledge of manifold theory, books such as [1] and [2] provide a good introduction. Additionally, the reader should have some knowledge of the axiom of choice and Zorn's lemma which are both briefly introduced in [3]. Furthermore, for further topics we touch on that are not relevant enough to be included in the thesis, we created two brief appendices. Appendix A gives a definition of vector bundles, whose sections, as we will see, are fine sheaves. On the other hand, Appendix B discusses some category theoretical concepts which we use throughout the thesis, of note we define the direct limit and show it preserves exactness.

2 Sheaves

2.1 Set-theoretical sheaves

In this section we introduce the notions presheaf and sheaf. We will also provide some examples and discuss sheafification. Most of the definitions lemmas and propositions are from Sheaf Theory by T.A. Springer [4] with a few adaptations by us. We begin by defining the presheaf:

Definition 2.1.1. Let X be a topological space. A **presheaf** $\mathcal{P}$ is an object that assigns to every open $U \subset X$ some set $\mathcal{P}(U)$ along with some maps $\rho_U^V : \mathcal{P}(V) \rightarrow \mathcal{P}(U)$ such that

- i. For an open $U \subset X$, $\rho_U^U = \text{id}_U$
- ii. For opens $U \subset V \subset W \subset X$, composition is transitive, that is, $\rho_U^V \circ \rho_V^W = \rho_U^W$

◇

So given a topological space X , a presheaf on X assigns data to opens in X in a manner consistent with some form of restriction maps. Here are some first examples:

Example 2.1.2. Let X be a topological space.

- An example is the presheaf of continuous functions. We define $\mathcal{P} : U \mapsto \mathcal{C}_X(U, \mathbb{R})$ and $\rho_U^V : \mathcal{C}_X(V) \rightarrow \mathcal{C}_X(U)$, $f \mapsto f|_U$ as the restriction of continuous functions on V to U . In this case the restriction maps are neither injective nor surjective in general. Take for example, $X = [0, 1] \subset \mathbb{R}$ with the subset topology, then clearly we can choose a restriction and piece-wise functions such that the restriction is not injective. Namely, consider the two functions 1 and

$$\begin{cases} 2x & x < 1/2 \\ 1 & x \geq 1/2 \end{cases}$$

then obviously restricting to $(1/2, 1]$ gives the same function. Now let $U = [0, 1/2)$ and $V = [0, 1]$, then the function $x \mapsto \frac{1}{x-1/2}$ is continuous on U yet not on V , hence ρ_U^V is not generally surjective.

- The constant presheaf is another example of importance later on. Let A be some set and let A_X be a presheaf on X where for any $U \subset X$ open $A_X(U) = A$ and the restriction maps are simply the identity on A .

△

We will not study presheaves independently in this thesis, rather, presheaves are a component of the definition of a sheaf.

Definition 2.1.3. Let $(U_i)_{i \in I}$ be an open cover of an open subset $U \subset X$, then a **sheaf** $\mathcal{S}$ is a presheaf such that:

- i. (Local) If $s, t \in \mathcal{S}(U)$ such that for all $i \in I$, $\rho_{U_i}^U(s) = \rho_{U_i}^U(t)$ then $s = t$. That is, when s, t coincide locally for all points $x \in U$, then $s = t$.
- ii. (Gluing) Let $(s_i)_{i \in I} \in \prod_{i \in I} \mathcal{S}(U_i)$ so that for all $i, j \in I$, $\rho_{U_i \cap U_j}^{U_i}(s_i) = \rho_{U_i \cap U_j}^{U_j}(s_j)$, in other words, so that s_i and s_j agree on their intersections. Then there exists an $s \in \mathcal{S}(U)$ such that $\rho_{U_i}^U(s) = s_i$ for all i . This s is called the **amalgamation**.

Let $\mathcal{T}$ be a sheaf over X such that for all $U \subset X$, $\mathcal{T}(U) \subset \mathcal{S}(U)$, we call $\mathcal{T}$ a **subsheaf** of $\mathcal{S}$. $\diamond$

As we can see, these properties are particularly useful in local to global problems. The locality axiom allows us to conclude that two object are equal if locally they are equal and the gluing axiom allows us to for example conclude some property holds on all of U if it holds locally on the U_i .

Example 2.1.4 (Sheaf of functions). Let X be a topological space. Denote the presheaf of functions by $\mathcal{F}$ with for any open $U \subset X$, $\mathcal{F}(U) := \{f : U \rightarrow \mathbb{R}\}$. Where for any $V \supset U$ open the restriction map is defined as $\rho_U^V(f)(x) = f(x)$ for any $x \in U$, so $\rho_U^V(f) = f|_U$. Now let $(U_i)_{i \in I}$ be a family of opens with union U .

First we prove $\mathcal{F}$ is local. If $f, g \in \mathcal{F}(U) : \rho_{U_i}^U(f) = \rho_{U_i}^U(g)$ for every $i \in I$, then in particular $f(x) = g(x)$ for every $x \in U$. Hence clearly $f = g$.

Next we prove glueability. Let $f_i \in \mathcal{F}(U_i)$ such that $\rho_{U_i \cap U_j}^{U_i}(f_i) = \rho_{U_i \cap U_j}^{U_j}(f_j)$ for all $i, j \in I$ (in words, the f_i are the same on their intersections). We define the amalgamation $f : U \rightarrow \mathbb{R}$ by, $f(x) = f_i(x)$ for $x \in U_i$. Clearly f is well-defined since if $x \in U_i \cap U_j$ for some sufficient i, j , then $f_i(x) = f_j(x)$.

In particular, the continuous presheaf mentioned in Example 2.1.2 is also a sheaf. This follows from the above argument, as well as from the pasting lemma. $\triangle$

Example 2.1.5 (Non-examples). Obviously not all presheaves are sheaves, here are two nonexamples:

- Take, for example, the presheaf $\mathcal{P}(U) := \mathcal{C}_X(U \times U)$ with restriction map the restriction on functions $\rho_U^V(f) = f|_{U \times U}$, $f \in \mathcal{P}(V)$. Neither properties of the sheaf hold. Namely, for a family of opens $(U_i)_{i \in I}$ with union $U \neq X$ the equality $\cup_{i \in I} U_i \times U_i = U \times U$ does not generally hold. Since this is the case, the locality and gluing properties cannot generally hold true.
- Another non-example of a sheaf is the constant presheaf A_X from example 2.1.2. Namely, let $U_1, U_2 \subset X$ be opens and $s_1 \in A_X(U_1) = A$ and $s_2 \in A_X(U_2) = A$. Suppose s is their amalgamation. Then $\rho_{U_2}^{U_1 \cup U_2}(s) = \text{id}_A(s) = s_1$ and $\rho_{U_1}^{U_1 \cup U_2}(s) = \text{id}_A(s) = s_2$ so $s_1 = s_2$ which is not generally true if $|A| > 1$.

$\triangle$

However, as we will see, we can generate a sheaf from every presheaf that is unique up to isomorphism. To accomplish this we will want to partition the set of all elements that the presheaf maps to by how they relate locally. Defining the equivalence relation that achieves this is a straightforward way of constructing a sheaf from a presheaf as we will see in Example 2.1.7.

Definition 2.1.6. Let $\mathcal{P}$ be a presheaf on X and let $x \in X$. The **stalk** of $\mathcal{P}$ at x is defined as

$$\mathcal{P}_x := \{(s, U) : U \subset X \text{ open, } x \in U, s \in \mathcal{P}(U)\} / \sim.$$

Where for $s \in \mathcal{P}(U), v \in \mathcal{P}(V)$, s and v are equivalent if, and only if, there exists an open neighborhood W of x such that $W \subset U \cap V$ and $\rho_W^U(s) = \rho_W^V(v)$. We denote $[s]_x$ as the equivalence class represented by $s \in \mathcal{P}(U)$ such that $x \in U$. This is written compactly in terms of the direct limit (see Appendix B), where the direct system is partially ordered by reverse inclusion, as,

$$\mathcal{P}_x := \varinjlim_{x \in U} \mathcal{P}(U).$$

$\diamond$

Example 2.1.7 (Godement Sheaf). Consider a set X endowed with the discrete topology and let $\mathcal{P}$ be a presheaf of X . Then the map $G_{\mathcal{P}} : U \mapsto \prod_{x \in U} \mathcal{P}_x$ defines a sheaf, called the Godement sheaf associated with the presheaf $\mathcal{P}$. This clearly defines a presheaf with for any pair of open sets $U, V \subset X$ the map $\rho_U^V : \prod_{x \in V} \mathcal{P}_x \rightarrow \prod_{x \in U} \mathcal{P}_x$. Now to prove this is a sheaf. Consider an open family $\{U_i\}_{i \in I}$ with union U and let $s, t \in G_{\mathcal{P}}(U)$ such that for all $i \in I$, $\rho_{U_i}^U(s) = \rho_{U_i}^U(t)$. Then in particular for any $x \in U$ there exists an $i \in I$ such that $x \in U_i$ and hence $s_x = t_x$ (where here $(\prod_{x \in U} \mathcal{P}_x)_x = \mathcal{P}_x$). Hence $s = t$. Now consider $s_i \in G_{\mathcal{P}}(U_i)$ such that $\rho_{U_i \cap U_j}^{U_i}(s_i) = \rho_{U_i \cap U_j}^{U_j}(s_j)$ for all $i, j \in I$. We want to show this has an amalgamation s . We know that for any $i, j \in I$, for all $x \in U_i \cap U_j$, $(s_i)_x = (s_j)_x$. Hence assuming the axiom of choice we can construct s by choosing for the x -th component representative s_x of the equivalence class $(s_i)_x$ where $i \in I : x \in U_i$, hence giving us an amalgamation $s := \prod_{x \in U} s_x$. We make no further mention of the axiom of choice in this thesis, assuming the spaces we work in are sufficiently simple. $\triangle$

A natural question we may have is whether or not there is a way to construct sheaves from presheaves. The answer to this question is yes and as it turns out there is such an operation defined by a universal property.

2.1.1 Sheafification: Constructing sheaves from presheaves

To be able to discuss this operation we first need to define the categories, that is the objects and morphisms. By considering the presheaves over a fixed topological space X , we can define the morphisms as follows:

Definition 2.1.8 (Morphism of presheaves of sets). For two presheaves on X , S, T a function $\varphi : S \rightarrow T$ such that for all opens $U \subset X$ $\varphi(U) : S(U) \rightarrow T(U)$ is a morphism of presheaves when the maps ρ commute with φ . That is given opens $U \subset V$ and the restriction map ρ_U^V in S , $\rho'_U{}^V$ in T the following diagram commutes,

$$\begin{array}{ccc} S(U) & \xrightarrow{\varphi(U)} & T(U) \\ \rho_U^V \downarrow & & \downarrow \rho'_U{}^V \\ S(V) & \xrightarrow{\varphi(V)} & T(V) \end{array}$$

$\diamond$

Of course, as with any category, we want to ensure compatible morphisms can be composed.

Remark 2.1.9 (Composition of morphisms). We can easily prove the composition. Given presheaves $S, T, \mathcal{P}$ on X and morphisms $f : S \rightarrow T$ and $g : T \rightarrow \mathcal{P}$, $g \circ f$ must also be a morphism. Clearly $g \circ f : S \rightarrow \mathcal{P}$ is a map. Let $U \subset X$ open, we define $g \circ f(U) := g(U) \circ f(U)$ such that $g \circ f(U) : S(U) \rightarrow \mathcal{P}(U)$. So clearly the first property holds. Let $V \subset U$ open. Note that since each of the two squares commutes in the following diagram, the diagram commutes.

$$\begin{array}{ccccc} S(U) & \xrightarrow{f(U)} & T(U) & \xrightarrow{g(U)} & \mathcal{P}(U) \\ \rho_U^V \downarrow & & \downarrow \rho'_U{}^V & & \downarrow \rho''_U{}^V \\ S(V) & \xrightarrow{f(V)} & T(V) & \xrightarrow{g(V)} & \mathcal{P}(V) \end{array}$$

Thus suitable morphisms are composable. $\diamond$

Definition 2.1.10. A morphism of presheaves $\varphi : S \rightarrow T$ is said to be an **isomorphism** if there exists a morphism $\psi : T \rightarrow S$ such that $\varphi \circ \psi$ is the identity morphism on T , which for all $U \subset X$ open and $s \in T(U)$ is defined as $\text{id}_T(U)(s) = s$ and $\psi \circ \varphi = \text{id}_S$. $\diamond$

Remark 2.1.11. Throughout the thesis we may denote the set of $\mathcal{C}$ -valued sheaves over X as $\mathbf{Sh}_{\mathcal{C}}(X)$. For examples, we will refer to the set of sheaves of $\mathcal{R}$ -modules on X by $\mathbf{Sh}_{\mathcal{R}\text{-mod}}(X)$. $\diamond$

Note that morphisms, and hence also isomorphism, of sheaves are the same as morphisms and isomorphisms of presheaves, only then between sheaves. In particular, for a isomorphism of presheaves, $\varphi : S \rightarrow T$ if S is a sheaf, then T is also. With this in mind, we can associate every presheaf to a sheaf. This is particularly useful in situations where we want to construct sheaves, for example when considering the image sheaf of a morphism of sheaves. The following proposition gives one such construction, using the Godement sheaf, we will provide a detailed proof of this statement and its associated universal property here.

Proposition 2.1.12 (Sheafification, [4], Proposition 2). *For a presheaf $\mathcal{P}$ on X there exist a pair $(F\mathcal{P}, \alpha)$ consisting of a sheaf $F\mathcal{P}$ and morphism α such that for every sheaf S and morphism $\beta : \mathcal{P} \rightarrow S$ there is a unique morphism $\gamma : F\mathcal{P} \rightarrow S$ such that $\beta = \gamma\alpha$.*

$$\begin{array}{ccc}
 \mathcal{P} & \xrightarrow{\beta} & T \\
 \alpha \downarrow & \nearrow \exists! \gamma & \\
 F\mathcal{P} & &
 \end{array}$$

For completeness we write out the details of the proof outlined in [4].

Proof. We define

$$F\mathcal{P}(U) := \{g \in G_{\mathcal{P}}(U) : \forall x \in U \exists \text{ open neighborhood } U' \subset U \text{ of } x \text{ and } s \in \mathcal{P}(U') : \forall y \in U' g_y = \rho_y^{U'}(s)\}.$$

In words, $g \in F\mathcal{P}(U)$ if $g \in G_{\mathcal{P}}(U)$ such that for all $x \in U$ there exists an open neighborhood of x , $U' \subset U$ and $s \in \mathcal{P}(U')$ such that for all $y \in U'$ $g_y = \rho_y^{U'}(s)$. So $F\mathcal{P}(U)$ is a subset of the godement sheaf $G_{\mathcal{P}}(U)$ (defined in Example 2.1.7) such that locally, in some neighborhood U' , the elements are the projection of some element in the presheaf $\mathcal{P}(U')$ onto the stalk.

$F\mathcal{P}$ is a sheaf and α is a morphism such that for every open $U \subset X$, $\alpha(U) : \mathcal{P}(U) \rightarrow F\mathcal{P}(U), s \mapsto \prod_{x \in U} \rho_x^U(s)$

The fact that $F\mathcal{P}$ is a presheaf is clear, the restriction maps can be defined by means of those from the Godement sheaf. Now we prove it is a sheaf. Let $(U_i)_{i \in I}$ be an open family with union U . Locality follows by a similar arguments as for the Godement sheaf. The construction of s for the amalgamation property is identical, we need only to show $s \in F\mathcal{P}(U)$. Let $x \in U$ then there exists an $i \in I$ such that $x \in U_i$. Take the $s_i \in F\mathcal{P}(U_i)$ from the construction clearly for all $y \in U_i$ there exists an open neighborhood $y \in U' \subset U$ and a $t \in \mathcal{P}(U')$ such that $t_y = s_{i_y} = s_y$ for all $y \in U' \cap U_i$ hence locally s is the image of some element in presheaf $\mathcal{P}$ thus $s \in F\mathcal{P}(U)$. So $F\mathcal{P}$ is a sheaf.

Now we show α is a morphism of presheaves. Let $U \subset X$ open and $s, s' \in \mathcal{P}(U)$ since $\alpha(s)_x = \rho_x^U(s)$ for all $x \in U$ (so $\alpha(s) \in F\mathcal{P}(U)$) and $s = s'$ implies that $\rho_x^U(s) = \rho_x^U(s')$ hence $\alpha(s) = \alpha(s')$. Thus α is well-defined. Now let $U \subset V \subset X$ opens, $\rho_U^V, \rho_U'^V$ be restrictions in $\mathcal{P}, F\mathcal{P}$ respectively. For any $s \in \mathcal{P}(V)$, $\alpha \circ \rho_U^V(s) = \Pi_{x \in U} \rho_x^U(\rho_U^V(s))$. Now $\rho_x^U(\rho_U^V(s)) = \rho_x^V(s)$ since $s \sim \rho_U^V(s)$ by definition and therefore

$$\Pi_{x \in U} \rho_x^V(s) = \rho_U'^V(\Pi_{x \in V} \rho_x^V(s)).$$

Consequently,

$$\alpha(U)(\rho_U^V(s)) = \Pi_{x \in U} \rho_x^U(\rho_U^V(s)) = \rho_U'^V(\Pi_{x \in V} \rho_x^V(s)) = \rho_U'^V(\alpha(V)(s))$$

so α is a morphism.

For any sheaf S and morphism $\beta : \mathcal{P} \rightarrow S$ there is a unique morphism $\gamma : F\mathcal{P} \rightarrow S$

Let S be a sheaf and $\beta : \mathcal{P} \rightarrow S$ a morphism. We prove existence of γ . We define $\alpha_S : S \rightarrow FS$ with $\alpha_S(U) : s \mapsto \Pi_{x \in U} \rho_x^U(s)$. We note that α_S is an isomorphism iff S is a sheaf (we outline the proof in 2. in Remark 2.1.13).

Now let us define a morphism $\varphi : F\mathcal{P} \rightarrow FS$ such that $\gamma = \alpha_S^{-1} \circ \varphi$ (where α_S^{-1} is an isomorphism by the previous paragraph). Consider any $s \in F\mathcal{P}(U)$. Let $x \in U$, by definition of $F\mathcal{P}$ there exists an open neighborhood $U_x \subset U$ of x and an $s_x \in \mathcal{P}(U_x)$ such $\rho_{U_x}^U(s) = s_x$. Now let $s = \Pi_{x \in U} [s_x]_x$ such that, $\varphi(U) : \Pi_{x \in U} [s_x]_x \mapsto \Pi_{x \in U} [\beta(s_x)]_x$. Indeed

$$\alpha_S(\beta(s)) = \Pi_{x \in U} [\beta(s)]_x = \varphi \circ \alpha(s).$$

Now to prove γ is a morphism of presheaves, since α_S^{-1} is a morphism we need only to show that φ is one.

Since β is a morphism for $t \in \mathcal{P}(V)$ we have that $s_x \sim t$ if, and only if, $\beta(s_x) \sim \beta(t)$ as the first equivalence is true if, and only if, there exists an open $W \subset U_x \cap V$ such that $\rho_W^{U_x}(s) = \rho_W^V(t)$ and β commutes with the restriction maps. Hence φ is a well-defined map. Clearly $\varphi(U)$ maps $F\mathcal{P}(U)$ into $FS(U)$ since for all $x \in U$ there exists an open neighborhood of x , $U' \subset U$ and $s \in \mathcal{P}(U')$ such that for all $y \in U'$, $[s_y]_y = \rho_y^{U'}(s)$ by definition of $F\mathcal{P}(U)$. Now under φ , $[\beta(s_y)]_y = \rho_y^{U'}(\beta(s))$ so $\varphi(U)(F\mathcal{P}(U)) \subset FS(U)$. From this we also obtain that φ commutes with restriction maps, so φ is a morphism.

Finally we prove the uniqueness of γ . Since α_S is an isomorphism this amounts to the uniqueness of φ . Let $\varphi' : F\mathcal{P} \rightarrow FS$ be any morphism such that $\varphi' \circ \alpha_S = \alpha_S^{-1} \circ \beta$. Let $s \in F\mathcal{P}(U)$ for any $x \in U$ there exists an open neighborhood of x , $U'_x \subset U$ and $s' \in \mathcal{P}(U'_x)$ such that $s_y = \rho_y^{U'_x}(s') = \alpha(s')_y$ for all $y \in U'_x$. Since φ' is a morphism,

$$\rho_{U'_x}^{U'}(\varphi'(s)) = \varphi'(\rho_{U'_x}^U(s)) = \varphi' \circ \alpha(s') = \alpha_S^{-1} \circ \beta(s') = \varphi \circ \alpha(s') = \rho_{U'_x}^U(\varphi(s)).$$

Now since $(U'_x)_{x \in X}$ is an open family with union U , by property i. of FS $\varphi(s) = \varphi'(s)$. Proving the claim. $\square$

Later on we will discuss sheaves over different categories than sets. It is important to note that this universal property does not immediately extend to sheaves over any category. Later on when discussing sheaves over modules we will have to verify this does indeed hold.

Remark 2.1.13. Sheaves and the sheafification have several interesting properties of note.

1. The pair $(F\mathcal{P}, \alpha)$ in Proposition 2.1.12 is unique up to isomorphism. This is a direct consequence of the universal property of sheafification. Let $F'\mathcal{P}$ be a sheaf and $\alpha' : \mathcal{P} \rightarrow F'\mathcal{P}$ a morphism such that these satisfy the universal property for sheafification. Choose first, $\beta = \alpha'$ and $T = F'\mathcal{P}$, then there is a unique $\gamma : F\mathcal{P} \rightarrow F'\mathcal{P}$ that is a morphism of sheaves by the universal property. Now, on the other hand, since $(F'\mathcal{P}, \alpha')$ also satisfies the universal property we can choose $\beta = \alpha$ and $T = F\mathcal{P}$ to obtain morphism $\gamma' : F'\mathcal{P} \rightarrow F\mathcal{P}$. Now, $\gamma' \circ \gamma : F\mathcal{P} \rightarrow F\mathcal{P}$ is the unique map satisfying the universal property for given $(F\mathcal{P}, \alpha)$, yet the map $\text{Id}_{\alpha_S(S)}$ also satisfies the property, thus by uniqueness $\gamma' \circ \gamma = \text{Id}_{\alpha_S(S)}$. Likewise $\gamma \circ \gamma' = \text{Id}_{\alpha'_S(S)}$, therefore, γ is sheaf isomorphism. The α is determined by $F\mathcal{P}$, as consequence the pair is unique up to isomorphism. This proof employs a standard technique used in category theory, this argument was written by the author with inspiration from an argument in Topologie and Meetkunde of another statement.
2. $\alpha_S : S \rightarrow FS$ is an isomorphism iff S is a sheaf. That S is a sheaf given isomorphism α_S follows quite quickly by the sheaf axioms. The converse follows quite easily from the universal property, as pointed out by my supervisor. Consider the following diagram,

$$\begin{array}{ccc} S & \xrightarrow{\text{id}} & S \\ \alpha \downarrow & \nearrow \gamma & \\ FS & & \end{array}$$

we see there exists a γ such that $\gamma \circ \alpha$ is the identity on S . Additionally, we may also conclude by the first diagram that,

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & FS \\ \alpha \downarrow & \nearrow \alpha \circ \gamma & \\ FS & & \end{array}$$

commutes. Now by the universal property, since replacing $\alpha \circ \gamma$ by the identity also makes this diagram commute, $\alpha \circ \gamma = \text{id}$.

◇

By the above remark, given a morphism φ in the category of sheaves the above proposition allows us to construct a morphism $F\varphi : FA \rightarrow FB$ that is unique up to isomorphism. This leads to the following result.

Corollary 2.1.14. *F is a functor between the category of presheaves and sheaves. That is, given any presheaf $\mathcal{P}$, $F\mathcal{P}$ is a sheaf and we can also sheafify any morphism of presheaves $\varphi : S \rightarrow T$ by F to obtain $F\varphi : FS \rightarrow FT$, in a manner that preserves identities and composition.*

Proof. In the universal property we saw that any presheaf $\mathcal{P}$ is sent to sheaf $F\mathcal{P}$. Hence we need to verify that we can F maps a morphism of presheaves to a morphism of sheaves. Let S, T be presheaves and $f : S \rightarrow T$ be a morphism. This induces morphism of sheaves,

$$Ff(U) : FS(U) \rightarrow FT(U), \Pi_{x \in U}[s_x]_x \mapsto \Pi_{x \in U}[f(s_x)]_x.$$

A trivial verification shows that this is indeed a morphism and that F maps identity functions to identity functions and preserves morphism composition. □

We determine this categorical relationship as it allows us to more easily construct sheaves for interesting objects we may want to study, as we will see when studying the kernel of sheaf morphisms. In this case,

as we will see it will be important to also be able to discuss the sheafification from another point of view. Therefore we will discuss a construction equivalent to the above in the sense that the two induced sheaves are isomorphic. For this construction we study geometric sheaves (étale spaces).

2.1.2 Geometric sheaves

The following definition and lemma are re-wordings of the constructions of the étale space given in [5] and [4]. We begin by defining a local homeomorphism:

Definition 2.1.15. A **local homeomorphism** is a map $\pi : A \rightarrow B$ where A, B are topological spaces such that for every $x \in A$ there exists an open neighborhood U_x of x such that $\pi|_{U_x} : U_x \rightarrow \pi(U_x)$ is a homeomorphism. $\diamond$

Example 2.1.16. Consider the two copies $S_1 \amalg S_1$ of some topological space S_1 and the map $\pi : S_1 \amalg S_1 \rightarrow S_1$ that canonically projects each component of $S_1 \amalg S_1$ onto S_1 . Clearly $\pi|_{(S_1, i)}$ for $i \in (0, 1)$ are homeomorphisms hence π is a local homeomorphism. In particular we see in this case that π is not injective. $\triangle$

Remark 2.1.17. Also any local homeomorphism maps opens to opens (is open). Namely, $\pi : A \rightarrow B$ be a local homeomorphism and $U \subset A$ open. For any $x \in U$, there is a U_x such that $\pi|_{U_x}$ is a homeomorphism. In particular, its restriction to $U \cap U_x$ is a homeomorphism. Now, $\cup_{x \in U} \pi(U \cap U_x) = \pi(\cup_{x \in U} U \cap U_x) = \pi(U)$. So since B is a topology $\pi(U) \subset B$ is open. As with local diffeomorphisms, if π is a bijective local homeomorphism, π is a homeomorphism. $\diamond$

Definition 2.1.18. Let $\pi : A \rightarrow B$ be a surjective local homeomorphism, then for an open $U \subset B$ a **section** is a continuous map $s : U \rightarrow A$ such that $\pi \circ s = \text{id}_U$. $\diamond$

This allows us to define the étale space of a sheaf to which we can associate a sheaf.

Definition 2.1.19. Let S be a presheaf on X . Define $S[X] := \coprod_{x \in X} S_x$ to be the topological space generated by topology basis $B(s, U) := \{[s]_x : x \in U\}$. $S[X]$ is called the **étale space** of S .

Then we define the **étale sheaf** associated to S by $S_e(U) := \{\tilde{s} : U \rightarrow S[X] : \tilde{s} \text{ section}\}$. $\diamond$

Additionally, the following properties hold for the particular topology chosen on $S[X]$.

Lemma 2.1.20. Let S be a presheaf on X . Then:

- i. $S[X]$ has the finest topology for which all the sections $\tilde{s} : U \rightarrow S[X], x \mapsto [s]_x$ are continuous.
- ii. The projection map $\pi : S[X] \rightarrow X$ is a local homeomorphism.

Proof. We prove the first claim and discuss the second,

- i. Let $\tilde{U} \subset S[X]$ and $[s]_x \in \tilde{U}$ (with $s \in S(U)$ for some open $U \subset X$), then for the section $\tilde{s}$ we find that $\tilde{s}^{-1}(\tilde{U}) = \{x \in U : [s]_x \in \tilde{U}\}$. So $\tilde{U}$ is open precisely when there is an open $U' \subset U : \tilde{s}^{-1}(\tilde{U}) = U'$, hence when $\tilde{U} = \{[s]_x : x \in U'\}$. So the topology for which all $\tilde{s}$ are continuous can contain any $B(s, U) = \{[s]_x : x \in U\}$ for any $U \subset X$ open and $s \in S(U)$. This is a basis of topology and hence we can construct a topology τ so that the property holds. All opens in τ can be characterized by the following property:

$\tilde{U} \in \tau \iff$ for every $[s]_x \in \tilde{U}$ there exists an open neighborhood $U \subset X$ of x such that $[s]_{x'} \in \tilde{U}$ for all $x' \in U$.

Suppose now there is a strictly finer topology $\tau \subset \tau'$ such that all these sections are continuous. Then there must exist a $C \in \tau'$ and a point $[s]_x \in C$ for which there is no open $U \subset X$ such that $\{[s]_x : x \in U\} \subset C$. In other words, $\tilde{s}^{-1}(C)$ is not open in X and hence $\tilde{s}$ is not continuous, a contradiction.

- ii. The following proof can also be found in [5]. The idea is that we show that the restriction of π to any element in the topology basis is a local homeomorphism. Let $U \subset X$ open and $s \in S(U)$ then for $\pi|_{B(s,U)}$, the inverse is given by $x \mapsto [s]_x$ so $\pi|_{B(s,U)}$ is a bijection. Now we characterize the opens in $B(s,U)$. Given $B(s,U), B(t,V)$ the intersection is given by $B(s,U) \cap B(t,V) = \{[s]_x : x \in W := \cup\{W' \subset U \cap V : W' \text{ open, } \rho_{W'}^U(s) = \rho_{W'}^V(t)\}\} = B(\rho_W^U(s), W) = B(s, W)$ (in words W is the largest open on which s and t coincide). Hence any open subset of $B(s,U)$ is of the form $B(s, V')$ for some $V' \subset U$ open. Consequently, for any open $V \subset B(s,U)$ there exists a $V' \subset X$ open so that $V = B(s, V')$ and hence $\pi|_{B(s,U)}(V) = V'$, thus the inverse is continuous. Hence π is a local homeomorphism. $\square$

The étale sheaf is indeed a sheaf and since we can construct morphism of presheaves $\alpha(U) : S(U) \rightarrow S_e(U), s \mapsto \tilde{s}$ (where $\tilde{s}$ is the section associated with s defined by $x \mapsto [s]_x$) for which the universal property described in Proposition 2.1.12 holds, we conclude by the uniqueness up to isomorphism argued in part 1 of Remark 2.1.13 that the two are isomorphic. This allows us to view a sheaf as both a sheaf and an étale sheaf. Furthermore,

Lemma 2.1.21. *The universal property described in Proposition 2.1.12 holds for étale sheaves.*

Proof. We will not prove this. A similar approach could be taken as with the other, constructing the maps α and γ and verifying these satisfy the universal property. See [6] for a proof. $\square$

Definition 2.1.22 (Constant and locally constant sheaf). Given a constant presheaf A_X , we denote the associated **constant sheaf** by $\underline{A}$ (e.g. given the constant presheaf $\mathbb{R}_X$, we obtain the constant sheaf $\underline{\mathbb{R}}$). Let S be a sheaf on X . This is said to be **locally constant** if for every $x \in X$ there exists a neighborhood U of x and constant sheaf $\underline{V}$ such that $S|_U$ is isomorphic to $\underline{V}$. Where here we consider the canonical restriction of S on X to $S|_U$ on U . $\diamond$

Example 2.1.23 (Constant sheaf). Let A be any set. Recall, we define the **constant presheaf** A_X on X as, $A_X(U) = A$ and $\rho_U^V = \text{Id}_A$ for $U \subset V$. For any $x \in X$ the stalk is given by

$$S_x = \{[s]_x : s \in C(U) \text{ for neighborhood } U \text{ of } x\} = \{[s]_x : s \in A\}$$

which is bijective to $\{x\} \times A$ since $s \sim t$ precisely when $s = t$ as elements of A . Hence since $\prod_{x \in U} S_x$ is bijective to $\prod_{x \in U} A$ the associated sheaf is (bijective to),

$$FC(U) = \{g \in \prod_{x \in U} A : \forall x \in U, \exists U' \subset U \text{ open neighborhood of } x, s \in C(U) = A : g_y = \rho_y^U(s) \forall y \in U'\}$$

since elements of $\coprod_{x \in U} A$ can be seen as maps $X \rightarrow A$ and any $g \in FC(U)$ is locally constant, the associated sheaf essentially maps U to the locally constant functions $U \rightarrow A$. These are two representations of the constant sheaf.

The associated etale sheaf

$$C_e(U) = \{\tilde{s} : U \rightarrow \prod_{x \in U} S_x : \tilde{s} \text{ section}\}$$

which is bijective to $C(U, A)$ since $\tilde{s}$ section and $\prod_{x \in U} S_x$ bijective to $X \times A$ implies that $\tilde{s}$ is in bijective correspondence with functions $f : U \rightarrow X \times A$ such that $\pi_1 \times f = \text{id}_U$ and f is any continuous function in the second coordinate (where A is endowed with the discrete topology, so this indeed amounts to the one obtained by sheafification). $\triangle$

Example 2.1.24 (Non-Hausdorff etale space). In general an etale space need not be Hausdorff. As also mentioned in lemma 1.1.30 of [6], and proven by identical means in a more general setting, given a smooth manifold M , the sheaf of smooth function, $C^\infty : U \mapsto C^\infty(U)$ for $U \subset M$ open, has a non-Hausdorff etale space. For some $p \in M$ the stalk C_p^∞ is the set of equivalence classes germs, where an equivalence class is of course the sets of functions that are equal in some neighborhood of p . Consider the smooth function $f \in C^\infty(M)$ whose set of nonzero points

$$U := \{x \in M : f(x) \neq 0\} \subsetneq M$$

is open, and the zero function denoted $0 \in C^\infty(M)$. Let $q \in \partial U$ and U_f, U_0 be open neighborhoods of f and 0 in the etale space $C^\infty[M]$. Since the projection map is a local homeomorphism (as posited in Lemma 2.1.20), we can choose these neighborhoods to be sufficiently small such that they are homeomorphic to open sets $\pi(U_f), \pi(U_0)$ in M . Since $f \sim 0$ in the stalk around q , $U_f \cap U_0 \neq \emptyset$ and hence

$$\pi(U_f) \cap \pi(U_0) = \pi(U_f \cap U_0) \neq \emptyset$$

is an open neighborhood of q . Thus there must exist a point $y \in \pi(U_f \cap U_0) \cap (M \setminus U)$, in which of course $[f]_y = [0]_y$. Now given the basis for topology of the etale space, any neighborhood of $[f]_q$ has non-trivial intersection with any neighborhood of $[0]_q$. Thus, $C^\infty[M]$ is not generally Hausdorff. $\triangle$

Now that we have considered first examples of working with sheaves we move on to defining some operations.

2.2 Sheaves of Modules

So far we only considered set-theoretical sheaves while in reality we will be interested in studying sheaves with some inherent algebraic structure. For example we may consider the sheaf of abelian groups. In this case a sheaf S on X is said to be a sheaf of abelian groups if for every $U \subset X$ open, $S(U)$ is an abelian group. In this case morphisms of sheaves $\varphi : S \rightarrow T$ are also required to be group homomorphisms as $\varphi(U) : S(U) \rightarrow T(U)$ for each $U \subset X$. We may also consider S to be a sheaf of rings in which case the $S(U)$ are rings and the morphisms are ring homomorphisms for every $U \subset X$. In this text we will be considering sheaves over algebraic objects as in the previous example where we considered the C^∞ sheaf which is a sheaf of rings as we will see. In this thesis we will make the choice of sheaves of modules which do implicitly require the definition of a sheaf of rings. For this we will follow the lecture notes of Andrew Lewis ([6]) as a reference.

Recall that a module is defined as follows,

Definition 2.2.1. Let $(R, +, \cdot)$ be a ring. A set M with the operations:

1. $+$ for which $(M, +)$ is an abelian group
2. $\times : R \times M \rightarrow M, (r, m) \mapsto r \cdot m$ an action on M such that for any $r, s \in R, m, n \in M$

(Distributive) $(r + s) \times m = r \times m + sm$ and $r \times (m + n) = r \times m + r \times n$

(Associative) $r \cdot s \times m = r \times (s \times m)$

Additionally if we further impose $1 \in R$,

(Unital) $1 \times m = m$

◇

Often we leave out the $\cdot$ and $\times$ in our computations. While technically this defines a left-module, we will be considering commutative rings and hence do not make the distinction. The morphisms in this category of R -modules are defined as:

Definition 2.2.2. Given R -modules M, N , an R -modules homomorphism is a map $\varphi : M \rightarrow N$ respecting the R -module structure, that is:

- For all $x, y \in M$, $\varphi(x + y) = \varphi(x) + \varphi(y)$
- For all $r \in R$ and $m \in M$, $\varphi(rx) = r\varphi(x)$

◇

Remark 2.2.3. We made the choice of sheaves of modules consciously. Recall that there is a 1-1 correspondence between $\mathbb{Z}$ -modules and abelian groups, that an R -module R is really a ring and if R is a field an R -module is a vector space. Hence defining the theory via modules, is a generalization in some sense. ◇

We can extend the notion of presheaf to rings, which we will need for presheaves of modules

Definition 2.2.4 (Presheaf of rings). Let X be a topological space. A presheaf of sets $U \mapsto \mathcal{R}(U)$ for which $\mathcal{R}(U)$ is a ring and the restriction maps are ring homomorphisms is a presheaf of rings. ◇

Example 2.2.5 (C_X^∞ as a presheaf of rings). Let X be a topological space. Remember that the restriction maps of C_X^∞ are just function restrictions. Given $U \subset X$, $C_X^\infty(U)$ has a ring structure given by pointwise addition and multiplication of functions. For $U \subset V \subset X$ opens and $f, g \in C_X^\infty(V)$, the restrictions have the ring homomorphism properties

$$\rho_U^V(f + g) = (f + g)|_U = f|_U + g|_U = \rho_U^V(f) + \rho_U^V(g)$$

and

$$\rho_U^V(fg) = fg|_U = f|_U g|_U = \rho_U^V(f) \rho_U^V(g).$$

△

Definition 2.2.6. Consider a presheaf of sets $\mathcal{M}$ over a topological space X and presheaf of rings $\mathcal{R}$. We call $\mathcal{M}$ a presheaf of $\mathcal{R}$ -modules if for every open $U \subset V \subset X$ $\mathcal{M}(U)$ is endowed with a $\mathcal{R}(U)$ -module structure such that the restriction map ρ_U^V of $\mathcal{M}$ is an $\mathcal{R}(V)$ -module homomorphism. That is the restriction map has the following properties:

1. ρ_U^U is the identity
2. For open $V \subset W$, $\rho_U^W = \rho_U^V \circ \rho_V^W$
3. Under $+$ the restriction map ρ_U^V is a group homomorphism
4. We have the following commutative diagram,

$$\begin{array}{ccc}
 \mathcal{R}(V) \times \mathcal{M}(V) & \xrightarrow{\quad \times \quad} & \mathcal{M}(V) \\
 \rho_U^V \downarrow & & \downarrow \rho_U^V \\
 \mathcal{R}(U) \times \mathcal{M}(U) & \xrightarrow{\quad \times \quad} & \mathcal{M}(U)
 \end{array}$$

such that $\mathcal{M}(U)$ becomes an $\mathcal{R}(U)$ -module with multiplication $(r, m) \mapsto \rho_U^V(r)m$.

◇

Definition 2.2.7 (Sheaf of modules). We say that a presheaf $\mathcal{M}$ of $\mathcal{R}$ -modules on X is a **sheaf of modules** if both $\mathcal{R}$ and $\mathcal{M}$ are sheaves in the sense of 2.1.3.

◇

Definition 2.2.8 (Morphisms of presheaves of modules). Let $\mathcal{M}, \mathcal{N}$ be presheaves of $\mathcal{R}$ -modules over X . A morphism $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ of $\mathcal{R}$ -modules is a morphism of presheaves of sets such that for any $U \subset X$ open $\varphi(U) : \mathcal{M}(U) \rightarrow \mathcal{N}(U)$ is an $\mathcal{R}(U)$ -module homomorphism.

◇

Remark 2.2.9. Similarly to morphisms of presheaves and sheaves of sets, a morphism of sheaves of $\mathcal{R}$ -modules is simply a morphism of presheaves of $\mathcal{R}$ -modules for which the domain and range are both sheaves of $\mathcal{R}$ -modules. Since we can compose module homomorphisms, we find by Remark 2.1.9 that morphisms of presheaves of modules can also be composed.

◇

Definition 2.2.10. Our notion of sheafification also holds for presheaves of rings and modules:

(Ring sheafification) Let $\mathcal{R}$ be a presheaf of rings. We define the sheafification of $\mathcal{R}$ as the sheafification $F\mathcal{R}$ for sets given some additional operations. Given $f, g \in F\mathcal{R}(U)$ we remember that for every $x \in U$ there exists a sufficiently small $V \subset U$ such that $f_x = [\tilde{f}]_x$ and $g_x = [\tilde{g}]_x$ for appropriate $\tilde{f}, \tilde{g} \in \mathcal{R}(V)$. We then define:

$$[\tilde{f}]_x + [\tilde{g}]_x := [\tilde{f} + \tilde{g}]_x, \quad [\tilde{f}]_x \cdot [\tilde{g}]_x := [\tilde{f} \cdot \tilde{g}]_x$$

Using the gluing property (since $F\mathcal{R}$ is a sheaf of sets) these operations can then be extended to the global setting on U . Thus giving $F\mathcal{R}(U)$ the ring structure. Using this we see that components-wise the restriction maps are ring homomorphisms. Namely let ρ_U^V be a restriction map of $F\mathcal{R}$ then it is the restriction $\rho_U^V|_{F\mathcal{R}(V)}$ of the restriction map ρ_U^V of the Godement sheaf $G_{\mathcal{R}}$ defined in Example 2.1.7, which preserves these newly defined ring operations components-wise. So $F\mathcal{R}$ is both a presheaf of rings and a sheaf in the sense of 2.1.3, therefore a sheaf of rings.

(Module sheafification) Let $\mathcal{R}$ be a presheaf of rings and $\mathcal{M}$ be a presheaf of $\mathcal{R}$ -modules. We define the sheafification of $\mathcal{M}$ as the sheafification $F\mathcal{M}$ for sets such that $F\mathcal{M}$ is a sheaf of $F\mathcal{R}$ -modules under the $F\mathcal{R}(U)$ -module operations defined components-wise on $F\mathcal{M}(U)$:

$$[\tilde{s}_x] + [\tilde{t}_x] := [\tilde{s} + \tilde{t}]_x, \quad [\tilde{f}_x] \times [\tilde{s}_x] = [\tilde{f} \times \tilde{s}]_x$$

Where $s, t \in F\mathcal{M}(U)$, $f \in F\mathcal{R}(U)$ and $x \in U$ and for some open $V \subset U$ sufficiently small such that there exist $\tilde{f} \in \mathcal{R}(V)$ and $\tilde{s}, \tilde{t} \in \mathcal{M}(V)$ such that $[\tilde{f}]_x = f_x$, $[\tilde{s}]_x = s_x$ and $[\tilde{t}]_x = t_x$. Under these operations the restriction maps ρ_U^V do indeed become $\mathcal{R}(V)$ -module homomorphisms. Thus by above $F\mathcal{R}$ is a sheaf of rings and $F\mathcal{M}$ is both a presheaf of modules and a sheaf in the sense of 2.1.3. Therefore $F\mathcal{M}$ is a sheaf of modules.

◇

The universal property of sheafification still holds for these sheafifications. For the proofs and generally more details on the constructions of sheaves of rings and modules we refer to [6]. Note however that sheafification there is defined by the more conventional approach via étale spaces. This is no problem though, as there is a 1-1 correspondence between the sheafification and étalification of a sheaf. As we will see when discussing operations on sheaves it is a useful tool to be able to interpret sheaves as étale sheaves. In fact, as done here, we could have immediately defined étale sheafification instead of sheafification.

Proposition 2.2.11 (Sheafification for modules). *Let $\mathcal{R}$ be a sheaf of rings and $\mathcal{M}$ be a presheaf of $\mathcal{R}$ -modules on X . There exist a pair $(F\mathcal{M}, \iota_{F\mathcal{M}})$ consisting of a sheaf of $\mathcal{R}$ -modules $F\mathcal{M}$ and morphism of $\mathcal{R}$ -modules $\iota_{F\mathcal{M}} : \mathcal{M} \rightarrow F\mathcal{M}$ such that for every sheaf $\mathcal{N}$ of $\mathcal{R}$ -modules and morphism $\psi : \mathcal{M} \rightarrow \mathcal{N}$ there is a unique morphism $\phi : F\mathcal{M} \rightarrow \mathcal{N}$ such that $\psi = \phi \circ \iota_{F\mathcal{M}}$.*

$$\begin{array}{ccc}
 \mathcal{M} & \xrightarrow{\psi} & \mathcal{N} \\
 \downarrow \iota_{F\mathcal{M}} & \searrow \exists! \phi & \\
 F\mathcal{M} & &
 \end{array}$$

Also the étale sheaf $\mathcal{M}_e$ and sheafification $F\mathcal{M}$ are isomorphic.

Proof. The universal property follows from Proposition 2.1.12 and the fact that we can show that all morphisms of sheaves of sets in the diagram are in fact morphisms of sheaves of modules, we will not prove this. Similarly Proposition 1.1.107 in [6] tells us the étale sheaf $\mathcal{M}_e$ (which is the étale space as the sheaf of sets with the module operations defined stalks-wise as in Definition 1.1.105 [6]) also satisfies the universal property, which proves the last claim. □

2.3 Some Examples: Sections of Vector Bundles

For a definition of vector bundles see Appendix A, we will consider smooth vector bundles in this section. The definition of smooth vector bundles enforces some smooth structure between neighboring vector spaces assigned via the vector bundle. For a given vector bundle $(E, M, \pi : E \rightarrow M)$, this allows us to define smooth maps between the vector spaces E_p given a sufficiently small neighborhood $U \subset M$, allowing us to consider the sections of a smooth vector bundle:

Definition 2.3.1. Given a smooth vector bundle $(E, M, \pi : E \rightarrow M)$ and some subset $U \subset M$, a **section** is a map $\sigma : U \rightarrow E$ that satisfies $\pi \circ \sigma = \text{id}_U$. ◇

Note that by definition a section is an injective smooth map. Also, given a section $\sigma : U \rightarrow E$, we have that $\sigma(p) \in E_p$ for any $p \in U$. In the case of the tangent bundle TM the sections are vector fields on M .

From every vector bundle, we can obtain a sheaf of sections. Namely given an open $U \subset M$, we define $\Gamma(-, E)$ as $\Gamma(U, E) := \{s : U \rightarrow E : s \text{ section}\}$.

Lemma 2.3.2. *Let (E, M, π) be a smooth $\mathbb{R}$ -vector bundle on M . Then $\Gamma(-, M)$ is a sheaf of C_M^∞ -modules on M .*

Proof. Proof by author, we use exercise 10.11 in [2] as inspiration. We claim that each $\Gamma(U, M)$ is a module. We know that $C_M^\infty(U)$ is a commutative ring under point-wise addition and multiplication. Also, since the composition of the function, $\varphi(p) = 0_{E_p}$, with a local trivialization is a function $p \mapsto (p, 0)$, φ is in fact smooth and a section, thus $\varphi = 0 \in \Gamma(U, M)$ so $\Gamma(U, M)$ has an additive identity. Under addition $\Gamma(U, M)$ is an closed, since given two $s, t \in \Gamma(U, M)$, $s(x) + t(x) \in E_x$ for all $x \in U$ as E_x is a vector space. Now $s + t : U \rightarrow E$ is a smooth function such that $\pi \circ (s + t) = \text{id}_U$ so $s + t \in \Gamma(U, M)$. The fact that addition is associative and commutative follows point-wise from the vector space structure. By a similar argument each element has an inverse. So $(\Gamma(U, M), +)$ is an abelian group.

Now let $U \subset M$ be open and let $\sigma, \tau \in \Gamma(U, M)$ and $f, g \in C_M^\infty(U)$. Now, $f \cdot \sigma + g \cdot \tau : M \rightarrow E$ is also smooth, where $\cdot$ is point-wise multiplication. It is also well-defined since for $p \in U$,

$$f(p)\sigma(p) + g(p)\tau(p) \in E_p$$

by the $\mathbb{R}$ -vector space structure of E_p . By this we also conclude that for any $p \in U$,

$$\pi(f(p)\sigma(p) + g(p)\tau(p)) = p$$

and thus

$$\pi(f \cdot \sigma + g \cdot \tau) = \text{id}_U.$$

The properties of the ring action on $\Gamma(U, M)$ follow by similar pointwise arguments.

By definition, $\Gamma(-, M)$ is a presheaf and since the sections are smooth functions, it follows by a similar argument as we used for the sheaf of continuous functions that $\Gamma(-, M)$ is a sheaf of sets. Now we prove it is a presheaf of modules. From the above each $\Gamma(U, M)$ is a module. Property i. and ii. follow immediately. The restriction maps are function restrictions in this setting, given sections $s, t : V \rightarrow E$ and $U \subset V$, clearly

$$\rho_U^V(s + t) = s|_U + t|_U = \rho_U^V(s) + \rho_U^V(t).$$

Furthermore, $\rho_U^V(s \times t) = \rho_U^V(s) \times \rho_U^V(t)$ giving us the desired $C^\infty(V)$ -module structure on $\Gamma(U, M)$. Hence, $\Gamma(-, M)$ is in fact a presheaf of C_M^∞ -modules. As a presheaf of sets, $\Gamma(-, M)$ a sheaf. Hence, $\Gamma(-, M)$ is sheaf of C_M^∞ -modules. $\square$

Interestingly, the sheaves of sections of the tangent and cotangent bundle relate to some important objects on smooth manifolds, namely the sheaves of vector fields and differential forms. Using the above lemma, we immediately obtain some of their important properties.

Example 2.3.3 (The tangent bundle). As we have mentioned previously the sections of the tangent bundle are vector fields on U . In fact there is a 1-1 correspondence between sections of the tangent bundle TM and vector fields on M . Let $U \subset M$ be open and denote the set of vector fields on U by $\mathfrak{X}(U)$.

Let $s : U \rightarrow TM$ be a section of TM . Denote $X_p := s(p)$, then by definition of π we find that $\pi(X_p) = \pi(s(p)) = p$ which implies that

$$X_p \in \pi^{-1}(p) = T_p M.$$

Since s is smooth $X : U \rightarrow TM$ smooth, so by definition $X \in \mathfrak{X}(U)$.

Now consider a vector field $X \in \mathfrak{X}(U)$. Define $s(p) := X_p$, then $\pi(s(p)) = \pi(X_p) = p$ since $X_p \in T_p M$. Also s is smooth since X smooth, hence $s : U \rightarrow TM$ is a section on U .

Thus by lemma 2.3.2, $\mathfrak{X}$ is a sheaf of C_M^∞ -modules. $\triangle$

Interestingly, by looking at the dual of the tangent space we obtain the sheaf of differential forms.

Example 2.3.4 (The cotangent bundle). Remember that given an $\mathbb{R}$ -vector space V we define its dual space V^* to be the set of linear functions $\omega : V \rightarrow \mathbb{R}$. In particular, given an m -dimensional manifold M , the dual to the tangent space $T_p M$ is the cotangent space $T_p^* M := (T_p M)^*$. We define $T^* M = \coprod_{p \in M} T_p^* M$ to be the cotangent bundle of M . Define the projection map as $\pi : T^* M \rightarrow M, (\omega, p) \mapsto p$. In appendix A we outline how such vector bundle operations can be defined, such that dualization. Thus by Lemma 2.3.2, given manifold M , the presheaf Ω^1 where $\Omega^1(U)$ is the collection of differential 1-forms on open subsets $U \subset M$ is a sheaf of C_M^∞ -modules.

Similarly, we can prove that Ω^k is a sheaf of C_M^∞ -modules by considering another vector bundle. Using the definition of wedge product define the k -wedge of $T^* M$ we can define $\bigwedge^k T^* M$ with a mapping,

$$\pi : \bigwedge^k T^* M \rightarrow M$$

such that the fibers are $\pi^{-1}(p) = \bigwedge^k T_p^* M$.

Now the wedge product can be described as an operation on vector bundles as described in Example ??, and so by this we can conclude that this is indeed a vector bundle. By lemma 2.3.2, we find that the sheaf of sections of $\bigwedge^k T^* M$ is a sheaf of C_M^∞ -modules. Moreover some verification shows that for an open $U \subset X$ the map $s : U \rightarrow \bigwedge^k T^* M$ is a section of $\bigwedge^k T^* M$ if, and only if, $s \in \Omega^k(U)$. From which we obtain the desired result that Ω^k is a sheaf of C_M^∞ -modules. $\triangle$

3 Cohomology

In this section we will define the notion of cohomology. In the process it is important that we work in a sufficiently "nice" category. For us, this will entail developing the ideas of kernel and cokernel, which is actually one of our primary reasons for considering sheaves of modules over sheaves of sets, since there is no obvious way to define the kernel of a morphism of a sheaf of sets, while for one of modules there is. In the first section we will discuss some operations of morphisms of sheaves of modules such as the kernel, cokernel, pushforward and pullback.

Remark 3.0.1. From now on, anywhere where we say sheaf we mean a sheaf of $\mathcal{R}$ -modules. Similarly by morphism we will mean a morphism of sheaves of $\mathcal{R}$ -modules. $\diamond$

3.1 Some operations on sheaves

We can now use the sheafification of presheaves of modules to construct some operations on these sheaves.

Definition 3.1.1 ([6], 1.3.6). Let $\mathcal{R}$ be a sheaf of rings $\mathcal{M}, \mathcal{N}$ be sheaves of $\mathcal{R}$ -modules over X . Then:

- i. We say $\mathcal{M}$ is a **subsheaf** of $\mathcal{N}$ if for every open $U \subset X$, $\mathcal{M}(U) \subset \mathcal{N}(U)$ and the inclusion $\iota : \mathcal{M} \rightarrow \mathcal{N}$ is a morphism of presheaves of $\mathcal{R}$ -modules. Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be a morphism of sheaves of modules.
- ii. The **image** of f , denoted $\text{im}(f)$ is the sheafification of the presheaf $U \mapsto \text{im } f(U)$.
- iii. The **kernel** of f , $\ker(f)$, is the sheafification of $U \mapsto \ker f(U)$.
- iv. If $\mathcal{M}$ is a subsheaf of $\mathcal{N}$, the **quotient** of $\mathcal{N}/\mathcal{M}$ is the sheafification of $U \mapsto \mathcal{N}(U)/\mathcal{M}(U)$.
- v. The **cokernel** of f , $\text{coker}(f)$. Is the quotient sheaf $\mathcal{N}/\text{im } f$.

The restriction maps of each of the presheaves is defined in the obvious way. $\diamond$

Note that by the definition of the subsheaf, the quotients taken in the quotient presheaf are meaningful since for a module M we can quotient out any submodule N (see Proposition 3 section 10.3 in [3]). For a more detailed definition of these operations see section 1.3.1 in [6].

Example 3.1.2. Consider the sheaves of commutative rings $C : U \mapsto C(U, \mathbb{R})$ and $C^\infty : U \mapsto C^\infty(U, \mathbb{R})$ with the the standard addition and multiplication of functions as operations. These can be seen as sheaves of C^∞, C -modules respectively. In both cases the restriction maps ρ_U^V are group homomorphisms as function restrictions (as mentioned in Example 2.2.5). We can view ρ_U^V as a $C^\infty(V), C(V)$ -module homomorphism by defined $\times : (f, g) \mapsto \rho_U^V(f)g = f_U g$ in the image. The inclusion $\iota : C^\infty \rightarrow C$ is clearly a module homomorphism for all $U \subset X$ and hence a morphism of C^∞ -modules.

(Subsheaf) For all $U \subset X$ note that $C^\infty(U)$ is a submodule of $C(U)$ hence C^∞ is a subsheaf of C via the inclusion.

(Kernel and image) The kernel and image are easily computed in this case, the image being C^∞ and kernel being the trivial sheaf of modules 0.

(Cokernel) The cokernel is the sheaf associated with $C(U)/C^\infty(U) = \{f \in C(U)\} / \sim$ where $f \sim g \iff f = c + g$ for some $c \in C^\infty(U)$. Since the equivalence classes are represented by continuous functions, this is in fact already a sheaf.

△

Lemma 3.1.3. *Let S, T be sheaves on X and $\alpha : S \rightarrow T$ a morphism. Then:*

1. $\ker(\alpha_x) = (\ker \alpha)_x$
2. $\text{im}(\alpha_x) = (\text{im}_{\text{pre}} \alpha)_x$

Proof. This follows from Proposition 1.3.2(i), (ii) from [6]. □

As a side note, in category theory, we may call functions monomorphic when they exhibit the properties of an injective function. In fact being a monomorphism is a generalization of being injective. In our setting, we may call a morphism of sheaves monomorphic when it behaves injectively. Similarly, when a morphism of sheaves behaves surjectively, we may call it an epimorphism, in particular:

Definition 3.1.4. Let $f : A \rightarrow B$ be a morphism of sheaves. Then we call f a **monomorphism** when $\ker f = 0$ and an **epimorphism** when $\text{im } f = B$. ◇

We now move on to defining the pushforward and pullback of a sheaf under a continuous function f , this will be useful later on in proving that cohomology is homotopy invariant.

3.1.1 Pushforward and pullback

We also have the notions of pushforward and pullback. The pushforward is fairly straightforward. It can be defined as follows,

Definition 3.1.5. Let X, Y be topological spaces and $f : X \rightarrow Y$ be a (continuous) map. The **pushforward** associated with the map f is denoted $f_* : \mathbf{Sh}_{\mathcal{R}\text{-mod}}(X) \rightarrow \mathbf{Sh}_{\mathcal{R}\text{-mod}}(Y)$ and maps the sheaf $\mathcal{M}$ of $\mathcal{R}$ -modules to the **pushforward sheaf** $U \mapsto \mathcal{M}(f^{-1}(U))$ for some open $U \subset Y$. The restriction maps of $f_*(\mathcal{M})$ are given by $f_*\rho_V^U = \rho_{f^{-1}(V)}^{f^{-1}(U)}$ for any $V \subset U$ open. ◇

Lemma 3.1.6. *Indeed the pushforward of a sheaf of sets is a sheaf of sets. More generally this also holds for modules.*

Proof. The first claim is Proposition 1.2.17 in [6]. By Proposition 1.1.12 [6], the pushforward of a presheaf of $\mathcal{R}$ -modules $\mathcal{M}$, $f_*(\mathcal{M})$, is a presheaf of $f_*(\mathcal{R})$ -modules, where $f_*(\mathcal{R})$ is a presheaf of rings by Proposition 1.2.8 [6]. Therefore, since both $f_*(\mathcal{M})$ and $f_*(\mathcal{R})$ are sheaves of sets, the pushforward of a sheaf of modules is a sheaf. □

The pullback however, is more technical. We will define it using the approach taken in [5]. For this we use the notion of pullback which is adjoint to the pushout, for details see the definition in appendix B.

Lemma 3.1.7 ([5], Lemma 3.1.1). *Let $p : E \rightarrow Y$ be a local homeomorphism and $f : X \rightarrow Y$ be a map of topological spaces. In the pullback diagram,*

$$\begin{array}{ccc}
 X \times_Y E & \xrightarrow{\pi_2} & E \\
 \pi_1 \downarrow & & \downarrow p \\
 X & \xrightarrow{f} & Y
 \end{array}$$

π_1 is a local homeomorphism.

Proof. Proof written by author. In this category, the pullback is concretely given by the tuple $(X \times_Y E, \pi_1 : X \times_Y E \rightarrow X, \pi_2 : X \times_Y E \rightarrow E)$ where π_1, π_2 are the canonical projections and $X \times_Y E = \{(x, e) \in X \times E : f(x) = p(e)\}$ (see Appendix B for the category theoretical definition).

Suppose then that p is a local homeomorphism. Then for every $x \in E$ there exists an open neighborhood of x , $U_x \subset E$ such that the restriction $p|_{U_x}$ is a homeomorphism with its image. Let $U := \pi_2^{-1}(U_x) \subset X \times_Y E$ (open since π_2 projection). Define $\alpha = (\iota, p|_{U_x}^{-1} \circ f \circ \iota)$ then clearly the following diagram commutes,

$$\begin{array}{ccccc}
 & & & p|_{U_x}^{-1} \circ f \circ \iota & \\
 & & & \curvearrowright & \\
 \pi_1(U) & & & & E \\
 & \searrow \alpha & & \xrightarrow{\pi_2} & \\
 & X \times_Y E & & & \\
 & \downarrow \pi_1 & & & \downarrow p \\
 & X & \xrightarrow{f} & Y & \\
 & \uparrow \iota & & & \\
 & \pi_1(U) & & &
 \end{array}$$

Note also that for any $x \in U$, by commutativity $\iota(\pi_1(x)) = \pi_1(\alpha(\pi_1(x)))$. So since $\alpha(\pi_1(x)) = x'$ for some $x' \in X \times_Y E$, $\pi_1(\alpha(\pi_1(x))) = \pi_1(x') = \pi_1(x)$. Also by commutativity,

$$\pi_2(x') = p|_{U_x}^{-1} \circ f(\pi_1(x)) = p|_{U_x}^{-1} \circ p \circ \pi_2(x) = \pi_2(x)$$

where the second equality holds from commutativity of the original diagram. Thus $x = x'$. Therefore $\alpha \circ \pi_1|_U = \text{id}_U$. So $\pi_1|_U$ homeomorphism. Since for any $y \in X \times_Y E$ there is a corresponding $x \in E$, π_1 is a local homeomorphism. $\square$

Now using this lemma we define the pullback as follows.

Definition 3.1.8. Let B be a sheaf on Y and $\pi : \text{Et}(B) \rightarrow Y$ be the associated étale map. Given a function $f : X \rightarrow Y$ of topological spaces its pullback is,

$$\begin{array}{ccc}
 X \times_Y \text{Et}(B) & \xrightarrow{\pi_2} & \text{Et}(B) \\
 \downarrow \pi_1 & & \downarrow \pi \\
 X & \xrightarrow{f} & Y
 \end{array}$$

So by the above lemma, $\pi_1 : X \times_Y \text{Et}(B) \rightarrow X$ is étale. From this we define the pullback $f^*(B)(-) = \Gamma(-, X \times_Y \text{Et}(B))$ which is indeed a sheaf of $\mathcal{R}$ -modules, as an étalification of a presheaf or $\mathcal{R}$ -modules. $\diamond$

At first glance it may not be obvious that this corresponds to the sheafification of the pushforward. We can however verify that on the stalks $f^*(B)_x \cong B_{f(x)}$, an exercise in [5], indicating that this is indeed the pushforward sheaf.

Finally, we establish a natural bijective correspondence between maps $\varphi : B \rightarrow f_*A$ and $\psi : f^*B \rightarrow A$. This correspondence will come in handy later on.

Proposition 3.1.9 (Adjunction formula). *Let B be a sheaf on Y and A a sheaf on X , and let $f : X \rightarrow Y$ be a continuous map. There exists a bijective correspondence between functions, $\varphi : B \rightarrow f_*A$ and $\psi : f^*B \rightarrow A$.*

Proof. Proposition 3.1.2 in the lecture notes [5]. $\square$

Another unrelated but useful result is an analogue to the first isomorphism theorem for sheaves.

Lemma 3.1.10 (First isomorphism theorem). *Let S, T be sheaves on X and $\varphi : S \rightarrow T$ be a morphism. Then $S/\ker \varphi \cong \operatorname{im} \varphi$.*

Proof. Follows by the first isomorphism theorem for modules, proof by author. Let $U \subset X$ be open, then $\varphi(U)$ is a morphism of $\mathcal{R}$ -modules. By the first isomorphism theorem for modules, $\psi_U : S(U)/\ker(\varphi(U)) \rightarrow \operatorname{im}(\varphi(U))$ is an isomorphism of $\mathcal{R}$ -modules. Define $\tilde{\varphi}_{\text{pre}} := \varphi_U$, note that for any open neighborhood V of U $\psi_U \circ \rho_U^V = \rho_U^V \circ \psi_V$. So $\tilde{\varphi}_{\text{pre}}$ is a bijective morphism of the presheaves $U \mapsto S(U)/\ker(\varphi(U))$ and $U \mapsto \operatorname{im}(\varphi(U))$. By definition 3.1.1 these are the presheaves corresponding to sheaves $S/\ker \varphi$ and $\operatorname{im} \varphi$. Thus the sheafification $\tilde{\varphi} := F\tilde{\varphi}_{\text{pre}}$ is the desired isomorphism. $\square$

Now that we introduced some useful properties of morphisms, we move on to studying sequences of such morphisms.

3.2 Exact sequences

In developing cohomology theory exact sequences play an important role. By moving to exact sequences, we are able to accomplish more in proofs by reasoning with this exactness, see e.g. Theorem 3.4.6. By the end of this chapter we will have developed the idea of cohomology being a measure of the inexactness of a cochain complex, whatever this means. For this we begin by defining exactness.

Definition 3.2.1. Suppose we are given a sequence of module homomorphisms,

$$\dots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} A_{n+2} \xrightarrow{f_{n+2}} \dots$$

We say this sequence is **exact in** A_{n+1} when $\ker f_{n+1} = \operatorname{im} f_n$ and **exact** when it is exact in all A_n . $\diamond$

Remark 3.2.2. Exactness can similarly be defined in any other category where maps have kernels and images. $\diamond$

In the category of sheaves, this definition becomes a bit more technical.

Definition 3.2.3 (Exact sequence of sheaves). Suppose we are given a sequence of morphisms of sheaves that compose,

$$\dots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} A_{n+2} \xrightarrow{f_{n+2}} \dots$$

We say the sequence is **exact in** A_{n+1} if $\ker f_{n+1} = \iota(\operatorname{im} f_n)$, where $\iota : \operatorname{im} f_n \rightarrow A_{n+1}$ is the inclusion of the image sheaf into A_{n+1} induced by the canonical inclusion of the image presheaf. We call the sequence **exact** if it is exact in all A_n . We may also call this a long exact sequence. Similarly, a short exact sequence is a sequence of the form $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ $\diamond$

Note however that in the case of exact sequences of sheaves, the sheafification of the image sheaf need not be unique, only up to isomorphism. Fortunately this is not a problem in our setting, for this we use the reasoning from a math stackexchange post [7] (math.stackexchange.com). Namely, let us compute the image and kernel at A_n in the above sequence. Recall that the kernel presheaf, $\ker f_n$, is already a sheaf so uniquely determined. Consider the presheaf $\operatorname{im}_{\operatorname{pre}} f_{n-1}$ defined by $U \mapsto \operatorname{im}(f_{n-1}(U))$ and two sheafifications $\operatorname{im} f_{n-1}$ and $\operatorname{im} f'_{n-1}$ which are isomorphic via a unique $\varphi : \operatorname{im} f_{n-1} \rightarrow \operatorname{im} f'_{n-1}$. Since $\operatorname{im}_{\operatorname{pre}} f_{n-1}$ is a subpresheaf of A_n with inclusion $\iota : \operatorname{im}_{\operatorname{pre}} f_{n-1} \rightarrow A_n$, the sheafification functor induces inclusions $\kappa : \operatorname{im} f_{n-1} \rightarrow A_n$ and $\kappa' : \operatorname{im} f'_{n-1} \rightarrow A_n$, such that $\kappa \circ \iota_{\operatorname{im} f_{n-1}} = \kappa' \circ \iota_{\operatorname{im} f'_{n-1}}$. By proposition 2.2.11 we also find that $\iota_{\operatorname{im} f'_{n-1}} = \varphi \circ \iota_{\operatorname{im} f_{n-1}}$ since φ is induced by the bijective morphism of presheaves $\operatorname{id}_{\operatorname{im}_{\operatorname{pre}} f_{n-1}}$. Therefore, $\kappa \circ \iota_{\operatorname{im} f_{n-1}} = \kappa' \circ \varphi \circ \iota_{\operatorname{im} f_{n-1}}$ and so by the fact the uniqueness of the morphisms in the sheafification, $\kappa = \kappa' \circ \varphi$, so that $\operatorname{im} \kappa = \operatorname{im} \kappa'$ as desired. This proves uniqueness under the assumption of existence, the existence follows from Lemma ??

Lemma 3.2.4. *Let S, S', S'' be sheaves of $\mathcal{R}$ -modules over X . Then the following are equivalent:*

- i. *The sequence $0 \rightarrow S' \xrightarrow{\alpha} S \xrightarrow{\beta} S'' \rightarrow 0$ is exact*
- ii. *for all $x \in X$ the sequence is exact in the stalks, that is, $0 \rightarrow S'_x \rightarrow S_x \rightarrow S''_x \rightarrow 0$ is exact*

And imply:

- iii. *For any $U \subset X$ the sequence of $\mathcal{R}(U)$ -modules, $0 \rightarrow S'(U) \xrightarrow{\alpha(U)} S(U) \xrightarrow{\beta(U)} S''(U) \rightarrow 0$ is exact.*

Proof. The following proof is a spelled out version of a proof in [4] with a few adaptations.

First we should prove the equivalence. Suppose $0 \rightarrow S' \xrightarrow{\alpha} S \xrightarrow{\beta} S'' \rightarrow 0$ is exact, then for all $U \subset X$, the sequence

$$0 \rightarrow S'(U) \xrightarrow{\alpha} S(U) \xrightarrow{\pi} S(U)/\operatorname{im} \alpha(U) \rightarrow 0$$

is exact where $\pi : S(U) \rightarrow S(U)/\operatorname{im} \alpha(U), s \mapsto [s]$ is the projection onto the quotient. Also by assumption, $\beta : S \rightarrow S''$ is a surjective morphism of sheaves. So by the first isomorphism theorem for sheaves $S/\ker \beta \cong S''$ where $\ker \beta = \operatorname{im} \alpha$. Thus $S'' \cong S/\operatorname{im} \alpha$, so $(S/\operatorname{im} \alpha)_x \cong S''_x$. Since the direct limit of exact sequences is exact, the two short exact sequences,

$$\begin{array}{ccccccc} 0 & \longrightarrow & S'_x & \longrightarrow & S_x & \xrightarrow{f} \varinjlim_{x \in U} [S(U)/\operatorname{im} \alpha(U)]_x & \xrightarrow{g} 0 \\ & & & & & \downarrow \varphi & \\ & & & & & S'_x & \end{array}$$

$\varphi \circ f$ $g \circ \varphi$

are exact since φ being an isomorphism (as it is isomorphic stalks-wise) implies that exactness in $(S/S')_x$ is equivalent to exactness in S''_x .

Suppose $0 \rightarrow S'_x \xrightarrow{\alpha_x} S_x \xrightarrow{\beta_x} S''_x \rightarrow 0$ is exact in S_x for all $x \in X$. Then $\ker(\beta_x) = \operatorname{im}(\alpha_x)$. So by lemma 3.1.3, $\ker(\beta)_x = \operatorname{im}_{\operatorname{pre}}(\alpha)_x$ for all $x \in X$. Under the inclusion κ of $\operatorname{im} \alpha$ into S_x as defined above, $\operatorname{im}_{\operatorname{pre}} \alpha$ is a subpresheaf of $\kappa(\operatorname{im} \alpha)$ and hence $\ker \beta$ is a subsheaf of $\kappa(\operatorname{im} \alpha)$, since stalk-wise $\ker \beta_x \subset (\kappa(\operatorname{im} \alpha))_x$. Note also that $(\beta \circ \alpha)_x = \beta_x \circ \alpha_x$ by definition, so $\beta \circ \alpha = 0$ and thus $\operatorname{im}_{\operatorname{pre}} \alpha \subset \ker \beta$ so $\kappa(\operatorname{im} \alpha) \subset \ker \beta$ by the universal property (we can consider inclusion $\iota : \operatorname{im}_{\operatorname{pre}} \alpha \rightarrow \ker \beta$ and by uniqueness the obtained κ' must behave identically to κ under the inclusion into S).

Now let's prove the exactness implies iii.: If the sequence is exact then $\ker \alpha = 0$ is the trivial sheaf. Let $U \subset X$ open, this means that $(\ker \alpha)(U) = 0$ so that $0 \rightarrow S'(U) \xrightarrow{\alpha(U)} S(U)$ is exact (as $(\ker \alpha)(U) = \ker(\alpha(U))$)

as the kernel of a morphism is automatically a sheaf). By exactness, $\ker \beta = \operatorname{im} \alpha$ is the sheaf associated with the presheaf $U \mapsto \operatorname{im} \alpha(U)$. This is in fact already a sheaf of modules however, since $\ker \alpha = 0$ implies that $\alpha : S' \rightarrow \operatorname{im} \alpha$ is a bijective morphism and thus an isomorphism of sheaves. Therefore, we conclude that $\ker \beta(U) = \operatorname{im} \alpha(U)$ so the sequence is exact. $\square$

3.2.1 Some Properties of the Sheafified Image

The following lemmas give some important properties of the sheafified image. The exactness of a sequence will play an important role in proofs in this thesis, hence the following lemmas will be used repeatedly. This first lemma is an application of exercise 2(b) from chapter 2 [5]. The second is formulated by the author (although is really already used implicitly in the proof of i. of Lemma 3.2.4), and both are proven by the author.

Lemma 3.2.5. *Let X be a topological space. If $\mathcal{P}$ is a subpresheaf of a sheaf $\mathcal{M}$ of $\mathcal{R}$ -modules on X , then there exists an associated sheaf $\tilde{\mathcal{P}} \subset \mathcal{M}$ such that $\mathcal{P} \hookrightarrow \tilde{\mathcal{P}}$ and elements of $\tilde{\mathcal{P}}$ are locally in $\mathcal{P}$. That is for any $U \subset X$ open and any $p \in \tilde{\mathcal{P}}(U)$, there exists a neighborhood U_p such that $\rho_{U_p}^U(p) \in \mathcal{P}(U_p)$. In particular, for a morphism of sheaves $\varphi : A \rightarrow B$, elements of $\operatorname{im} \varphi$ are locally in the image presheaf, hence the image sheaf behaves locally like an image.*

Proof. We begin by defining the presheaf $\tilde{\mathcal{P}}$ as,

$$\tilde{\mathcal{P}}(U) := \{r \in \mathcal{M}(U) : \text{for all } x \in U \text{ there exists a neighborhood } W_x \subset U \text{ of } x \text{ such that } \rho_{W_x}^U(r) \in \mathcal{P}(W_x)\}.$$

The details are omitted. By some argumentation, we can show that $\tilde{\mathcal{P}}$ is a presheaf, the fact that $\tilde{\mathcal{P}}(U)$ is a module follows using the property involving the restriction maps ρ associated to $\mathcal{M}$ which are by assumption module homomorphisms. The restriction maps of $\tilde{\mathcal{P}}$ can be defined using those of $\mathcal{M}$. Additionally, we can argue this $\tilde{\mathcal{P}}$ is a sheaf.

That $\tilde{\mathcal{P}}$ is the associated sheaf entails showing that the universal property of sheafification holds, verifying this goes by an analogous procedure as in Proposition 2.1.12. Also by the definition of $\mathcal{P}$, inclusion $\mathcal{P} \hookrightarrow \tilde{\mathcal{P}}$ is canonical one and the elements of $\tilde{\mathcal{P}}$ are locally in $\mathcal{P}$. Proving the final claim amounts to the observation that $\operatorname{im}_{\text{pre}} \varphi$ is a subpresheaf of B . $\square$

Lemma 3.2.6 (Monomorphic morphisms have normal image). *Let $\varphi : A \rightarrow B$ be a monomorphic morphism of sheaves. For any $x \in \operatorname{im} \varphi(U)$, there exists a $y \in A(U)$ such that $\varphi(U)(x) = y$.*

Proof. By Lemma ??, locally x is in the image presheaf of φ , that is, for all $p \in U$ there exists an open neighborhood $U_p \subset U$ of p such that $\rho_{U_p}^U(x)$ is in the image presheaf of φ . Hence for all $p \in U$, there exists a $y_p \in A(U_p)$ such that $\varphi(U_p)(y_p) = \rho_{U_p}^U(x)$. Note that for any other $p' \in U$ such that $U_p \cap U_{p'} \neq \emptyset$,

$$\rho_{U_p \cap U_{p'}}^{U_p}(\varphi(U_p)(y_p)) = \rho_{U_p \cap U_{p'}}^U(x) = \rho_{U_p \cap U_{p'}}^{U_{p'}}(\varphi(U_{p'})(y_{p'})).$$

So since φ is a morphism of sheaves and $\varphi(U_p \cap U_{p'})$ is injective,

$$\rho_{U_p \cap U_{p'}}^{U_p}(y_p) = \rho_{U_p \cap U_{p'}}^{U_{p'}}(y_{p'})$$

for any such $p' \in U$. So since A is a sheaf we can apply the gluing axiom to obtain a $y \in A(U)$ such that $\rho_{U_p}^U(y) = y_p$. By the locality axiom, we then deduce that $\varphi(U)(y) = x$. $\square$

3.2.2 Examples

Now let us consider some interesting examples of exact sequences.

Example 3.2.7 (De Rham complex). Let $\underline{\mathbb{R}}$ be the constant sheaf of $\mathbb{R}$ -modules on an m -dimensional manifold M . We have shown that, since $\mathbb{R}$ is connected, $\underline{\mathbb{R}}$ is isomorphic to the sheaf of constant functions. Define a sheaf morphism $\epsilon : \underline{\mathbb{R}} \rightarrow \Omega^0, (U, r) \mapsto r \in C^\infty(U)$.

Recall also that locally any p -form ω can be expressed in terms of the basis induced by the chart. So that for chart (U, φ) , any $\omega \in \Omega^p(U)$ can be expressed as $\omega = \sum_{I \in \text{Ord}_p(M)} f_I \cdot (d\varphi)_I$, where $\text{Ord}_k(M)$ is the set of increasing k -tuples of indices, as defined in [1]. Using this information, we can define the morphisms $d^p : \Omega^p \rightarrow \Omega^{p+1}$ locally as,

$$d^p(U)(\omega) = \sum_{I \in \text{Ord}_{p+1}(M)} \left(\left(\sum_{i=1}^m \frac{\partial f_I}{\partial x_i} dx_i \right) \wedge (d\varphi)_I \right)$$

which are homomorphisms of $\mathbb{R}$ -modules. Namely, define for any $\omega \in \Omega^k(V)$, for $U \subset V \subset M$ opens, $\rho_U^V(d^k(V)(\omega))$ as $d^k(U)(\rho_U^V(\omega))$. For any $p \in V$ there exists a chart (U_p, φ_p) with $U_p \subset V$ such that,

$$\rho_p^V(d^k(V)(\omega)) = \rho_p^{U_p} \circ \rho_{U_p}^V(d^k(V)(\omega)) = \rho_p^{U_p} \circ d^k(U_p)(\rho_{U_p}^V(\omega)) = \left[\sum_{I \in \text{Ord}_p(M)} \left(\left(\sum_{i=1}^m \frac{\partial f_I}{\partial x_i} dx_i \right) \wedge (d\varphi)_I \right) \right]_p.$$

Hence for any open $U \subset M$ and section $\omega \in \Omega^k(U)$, we can obtain $d^p(U)(\omega)$ by means of an open cover of sets contained in domains of charts $\mathcal{U}$ of U using the above and locality of the sheaf Ω^{p+1} . It holds that $d^0 \circ \epsilon = 0 = d^{n+1} \circ d^n$ for all $n \in \mathbb{N}$, since this is true locally, and hence by the locality axiom, globally. In the next example, we will show that this sequence is locally exact, that is, for every $p \in M$ there is some neighborhood U_p so that the sequence evaluated in U_p is exact. The standard de Rham cohomology may fail to be exact, which correspond to the cohomology of the global sections of this complex. $\triangle$

Exact sequences are also a compact way to express some important properties a series of modules may have.

Example 3.2.8 (Poincaré lemma). Recall that the Poincaré lemma, Theorem 5.37 in [1], essentially says that given an m -dimensional manifold M on a neighborhood $U \subset M$ such that U is diffeomorphic to $\mathbb{R}^m$ any closed form on U of strictly positive degree is exact. This translates to the sequence,

$$0 \rightarrow \mathbb{R} \xrightarrow{\epsilon(U)} \Omega^0(U) \xrightarrow{d^0(U)} \Omega^1(U) \xrightarrow{d^1(U)} \dots$$

being exact. $\triangle$

Example 3.2.9 (Defining the sheaf via an exact sequence). This example is taken from [4]. Consider a presheaf of modules $\mathcal{P}$ on topological space X . Then $\mathcal{P}$ is a sheaf if, and only if, for any open $U \subset X$ with open cover $\{U_i\}_{i \in I}$ and sheaf morphisms,

$$\alpha : s \mapsto (\rho_{U_i}^U(s))_i, \quad \beta : (s)_i \mapsto \left(\rho_{U_k \cap U_l}^{U_k}(s_k) - \rho_{U_k \cap U_l}^{U_l}(s_l) \right)_{(k,l)}$$

the sequence,

$$0 \rightarrow \mathcal{P}(U) \xrightarrow{\alpha(U)} \prod_i \mathcal{P}(U_i) \xrightarrow{\beta(U)} \prod_{i,j} \mathcal{P}(U_i \cap U_j)$$

is exact. Here the injectivity of the $\alpha(U)$ enforce the locality property. Now given $s_i \in \mathcal{P}(U_i)$ for which the conditions for gluing hold, implies that $(s_i)_i \in \ker \beta(U)$. So by exactness, since $\text{im}(\alpha(U)) = \ker \beta(U)$, there exists an $s \in \mathcal{P}(U)$ that is in fact the amalgamation of the $(s_i)_i$ by definition of the $\alpha(U)$ so describes the gluing property. We can further check that the converse also holds. $\triangle$

Note that all the sequences we considered above have the property that any composition of adjacent morphisms is zero. A sequence with this property is called a cochain complex. In the next section we study these objects in depth to obtain some important properties.

3.3 Cochain complexes

In our definition of cohomology, cochain complexes play an important role. In this section we will cover cochain complexes and maps between them as well as the notion of cochain homotopy, which will play an important role in the proof that our definition of cohomology is meaningful. The definitions and results are inspired by those from a book on homological algebra by Weibel [8] and the lecture notes from the masters seminar on sheaf cohomology at Utrecht University from the '90s [5]. The author would suggest both of these sources as supplement.

Definition 3.3.1 (Cochain complex). A **cochain complex** $\mathcal{C}^\cdot$ of sheaves (of $\mathcal{R}$ -modules) on X is a countable family $\{C^i\}_{i \in \mathbb{Z}}$ of sheaves along with morphisms $d^n : C^n \rightarrow C^{n+1}$ such that $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$. Now given a cochain complex $\mathcal{C}^\cdot$:

(cocycles) The kernel of the n -th map d^n is the set of all n -**cocycles** denoted $Z^n(\mathcal{C}^\cdot) = \ker d^n$

(coboundaries) The image of the $(n-1)$ -th map d^{n-1} is the set of all n -**coboundaries** denoted $B^n(\mathcal{C}^\cdot) = \text{im } d^{n-1}$

(cohomology) We define the n -th cohomology module of $\mathcal{C}^\cdot$ as $H^n(\mathcal{C}^\cdot) := Z^n(\mathcal{C}^\cdot)(X)/B^n(\mathcal{C}^\cdot)(X)$. This is a meaningful definition since by definition of cochain complexes, $B^n(\mathcal{C}^\cdot)(X) \subset Z^n(\mathcal{C}^\cdot)(X)$ so a submodule, so the quotient is well-defined by Proposition 10.2.3 in Dummit and Foote [3].

(map of cochain complexes) Now given a second cochain complex $\mathcal{D}^\cdot$ of sheaves, we call a family of morphisms $u^n : C^n \rightarrow D^n$ a cochain map $u : \mathcal{C}^\cdot \rightarrow \mathcal{D}^\cdot$ if the following diagram commutes:

$$\begin{array}{ccccccc} \dots & \longrightarrow & C^{n-1} & \xrightarrow{d_C^{n-1}} & C^n & \xrightarrow{d_C^n} & C^{n+1} & \xrightarrow{d_C^{n+1}} & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & D^{n-1} & \xrightarrow{d_D^{n-1}} & D^n & \xrightarrow{d_D^n} & D^{n+1} & \xrightarrow{d_D^{n+1}} & \dots \end{array}$$

$\diamond$

Cochain maps can be added and subtracted, and composed. Let $\mathcal{C}^\cdot, \mathcal{D}^\cdot, \mathcal{E}^\cdot$ be cochain complexes and $f, g : \mathcal{C}^\cdot \rightarrow \mathcal{D}^\cdot$ and $h : \mathcal{D}^\cdot \rightarrow \mathcal{E}^\cdot$ be chain maps. For each of the operations it can easily be verified that the operations $(f \pm g)^n := f^n \pm g^n$, and $(h \circ f)^n := h^n \circ f^n$ give rise to new cochain maps.

We can speak of cochain complexes being bounded from above or below, these two notions correspond to the indexing set having a maximal or minimal element respectively. For example, the short exact sequence is an example of a cochain complex bounded in both directions. An important observation is that any morphism of cochain complexes induces a homomorphism in cohomology.

Lemma 3.3.2. *Any map of cochain complexes of sheaves $u : \mathcal{C} \rightarrow \mathcal{D}$ can be pushed forward to an $\mathcal{R}(X)$ -module homomorphism $u_*^n : H^n(\mathcal{C}) \rightarrow H^n(\mathcal{D})$.*

Proof. Proof by author. The reader can easily verify that a map of cochain complexes maps cocycles (coboundaries) to cocycles (coboundaries). Using this information we define for every $n \in I$,

$$u_*^n : (Z^n(\mathcal{C})(X)/B^n(\mathcal{C})(X)) = H^n(\mathcal{C}) \rightarrow H^n(\mathcal{D}), [g] \mapsto [u^n(X)(g)].$$

This map is well-defined since, $u^n \circ d_{\mathcal{C}}^{n-1} = d_{\mathcal{D}}^{n-1} \circ u^{n-1}$, $g \sim g'$ implies that $u^n(X)(g) \sim u^n(X)(g')$. The fact that this is an $\mathcal{R}(X)$ -module homomorphism follows from the definition of the quotient module. That is given $g, g' \in H^n(\mathcal{C})$ and $r \in \mathcal{R}(X)$:

$$u_*^n(g + rg') = [u^n(X)(g + rg')] = [u^n(X)(g) + ru^n(X)(g')] = [u^n(X)(g)] + r[u^n(X)(g)] = u_*^n(g) + ru_*^n(g')$$

where the second equality holds since $u^n(X)$ is an $\mathcal{R}(X)$ -module homomorphism and the third by definition of the quotient module. Thus by Proposition 10.2.2 (1) from Dummit and Foote, u_*^n is an $\mathcal{R}(X)$ -module homomorphism. $\square$

Lemma 3.3.3. *Given cochain maps $f, g : \mathcal{C} \rightarrow \mathcal{D}$, it holds that $(f \pm g)_* = f_* \pm g_*$*

Proof. Recall that every element in $H^n(\mathcal{C})$ is an equivalence class brought forth by some element $x \in \ker d_{\mathcal{C}}^n$. We know that by definition in lemma 3.3.2, $(f \pm g)_*([x]) = [(f \pm g)^n(X)([x])] = [(f^n \pm g^n)(X)([x])] = [f^n(X)([x]) \pm g^n(X)([x])]$. By definition of the quotient module this equals $[f^n(X)([x])] \pm [g^n(X)([x])] = f_*([x]) \pm g_*([x])$. Hence, $(f \pm g)_* = f_* \pm g_*$. $\square$

Definition 3.3.4 (quasi-isomorphism). A morphism $u : \mathcal{C} \rightarrow \mathcal{D}$ is said to be a **quasi-isomorphism** of cochain complexes if its pushforward induces isomorphism's in cohomology, $u_*^n : H^n(\mathcal{C}) \rightarrow H^n(\mathcal{D})$ for all $n \in I$. $\diamond$

Definition 3.3.5 (Acyclic cochain complex). A cochain complex is said to be **acyclic** if its cohomology groups are trivial. $\diamond$

Since a cochain complex is a sequence of morphisms we can also speak of its exactness.

Remark 3.3.6 (Equivalence of definitions). For $\mathcal{C}$ cochain complex following statements are equivalent:

- i. $\mathcal{C}$ is exact
- ii. $\mathcal{C}$ is acyclic
- iii. The cochain map $0 \rightarrow \mathcal{C}$ is a quasi-isomorphism (where 0 is the trivial cochain complex)

The equivalence of ii. and iii. is clear from the definitions since $0 \rightarrow \mathcal{C}$ is a quasi-isomorphism precisely when $0 = H^n(0) \rightarrow H^n(\mathcal{C})$ which is true if, and only if, $H^n(\mathcal{C}) = 0$ for all $n \in I$. The equivalence of i. and ii. follows since $\mathcal{C}$ exact if, and only if, $\ker d^n = \text{im } d^{n-1}$ if, and only if, $H^n(\mathcal{C}) = \ker d^n / \text{im } d^{n-1} = 0$ for all $n \in I$. So as noted at the beginning of this chapter, we can speak of cohomology being a measure of the extent to which a sequence is exact. $\diamond$

3.3.1 Cochain homotopy

Definition 3.3.7 (Cochain homotopy). Given two cochain maps $f, g : \mathcal{C} \rightarrow \mathcal{D}$ we say they are **cochain homotopic** if there exists a collection of morphisms $h^n : C^n \rightarrow D^{n-1}$ such that $f^n - g^n = d_D^{n-1} \circ h^n + h^{n+1} \circ d_C^n$. $\diamond$

Note that the maps h^n need not be chosen so that they commute in the diagram of cochain maps. Cochain homotopy is a kind of abstraction of homotopies between continuous maps. We find it important because of i. in the following proposition.

Proposition 3.3.8. *The following are true for cochain homotopies.*

i. For cochain homotopic maps $f, g : \mathcal{C} \rightarrow \mathcal{D}$, $f_* = g_*$

ii. Cochain homotopy is an equivalence relation

Proof. We prove the first statement by a similar argument as for Lemma 1.4.5 in [8]. Suppose that f, g are chain homotopic. Then for all n , $f^n - g^n = d_D^{n-1} \circ h^n + h^{n+1} \circ d_C^n$. Let $[x] \in H^n(\mathcal{C})$. By Lemma 3.3.3 and the definition of the pushforward of a cochain map (Lemma 3.3.2), $f_*([g]) - g_*([x]) = (f - g)_*([x]) = [(f^n - g^n)(x)] = [d_D^{n-1} \circ h^n(x) + h^{n+1} \circ d_C^n(x)] = [d_C^{n-1} \circ h^n(x)]$. So in particular, $(f^n - g^n)(x)$ is an n -coboundary in $\mathcal{D}$, so $[(f^n - g^n)(x)] = [0] = 0$. Thus $f_*([x]) - g_*([x]) = 0$. Since $[g]$ was an arbitrary element of $H^n(\mathcal{C})$, $f_* = g_*$.

The fact this is an equivalence relation follows easily from the definition and so is left to the reader. $\square$

3.4 Resolution via Injective sheaves

In this thesis being able to compute the cohomology of a manifold will be important. We do this via chain complexes as defined above. In particular we will compute cohomology via resolutions of the space,

Definition 3.4.1 (Resolution). Given a sheaf S , a **resolution of S** is a cochain complex $\mathcal{C}$ of sheaves such that

$$0 \rightarrow A \rightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \dots$$

is exact. $\diamond$

For proving statements later on about cohomology via a resolution it will be useful for the sheaves of $\mathcal{C}$ to have some inherent property. For us it will be useful to be able to extend the domain of a morphism being mapped into a given sheaf. This then will allow us to compute the cohomology of a topological space via a given sheaf S . The goal of this subsection will be to introduce the desired property; the injectivity of a sheaf.

Definition 3.4.2 (Injective sheaf). A sheaf C is **injective** if for any monomorphism of sheaves $\iota : A \hookrightarrow B$, a morphism $\varphi : A \rightarrow C$ can be extended via ι to a morphism $\tilde{\varphi} : B \rightarrow C$, so that the following diagram commutes,

$$\begin{array}{ccc} A & \xhookrightarrow{\iota} & B \\ \varphi \downarrow & \nearrow \tilde{\varphi} & \\ C & & \end{array}$$

$\diamond$

Note however an extension of a morphism φ via an ι may not be unique. We discuss later on why this is not an issue for computing cohomology by means of the comparison theorem proved at the end of this subsection.

Definition 3.4.3 (Injective Resolution). A resolution of a sheaf $\mathcal{C}$ in which each $\mathcal{C}^i$ is injective is called an **injective resolution**. $\diamond$

Obviously, to guarantee that every sheaf admits an injective resolution, every sheaf must be able to be embedded into an injective sheaf, this is often referred to as the category (in this case of sheaves) admitting enough injectives.

Proposition 3.4.4 (The category of sheaves admits enough injectives). [5] *For a sheaf $\mathcal{M}$ on X there exists a monomorphism into an injective sheaf $\mathcal{I}$.*

Proof. Proof of lemma 4.3.1 in [5] adjusted for sheaves of $\mathcal{R}$ -modules. We use the fact that for any ring R the category of R -modules admits enough injectives, this is theorem 10.5.38 in Dummit and Foote [3]. We alternatively defined the stalk of a sheaf $\mathcal{M}$ as $\mathcal{M}_x = \varinjlim_{U \ni x} \mathcal{M}(U)$ it can be shown that this admits an $\mathcal{R}_x$ -module structure (as mentioned in Appendix B). Thus the stalks $\mathcal{M}_x$ are $\mathcal{R}_x$ -modules and can be embedded into injectives $\mathcal{I}_x$ via $\sigma_x : \mathcal{M}_x \hookrightarrow \mathcal{I}_x$. We can construct the Godement sheaf $\mathcal{I}(U) := \prod_{x \in U} \mathcal{I}_x$ which we claim is our desired injective.

$\mathcal{M}$ can be embedded into $\mathcal{I}$ via the map $\sigma : \mathcal{M} \rightarrow \mathcal{I}$ that is defined stalks-wise as $(\sigma_U(m))_x = \sigma_x([m]_x)$ for all open neighborhoods of $x, U \subset X$ and elements $x \in \mathcal{M}(U)$. By a standard argument, using the fact the codomain is a Godement sheaf, we conclude σ defines a morphism. Similarly, the σ_x injective imply σ is injective. So $\mathcal{M} \hookrightarrow \mathcal{I}$.

We claim $\mathcal{I}$ is injective. Let $\varphi : \mathcal{P} \rightarrow \mathcal{I}$ be any sheaf map and $\iota : \mathcal{P} \hookrightarrow \mathcal{Q}$. By the above we see that φ is equivalently the collection of stalk maps $\varphi_x : \mathcal{P}_x \rightarrow \mathcal{I}_x$ (since the stalk maps can be obtained from φ and we can reconstruct φ from the stalk maps by similar means as we did for σ). Since the $\mathcal{I}_x$ are injective modules and ι induces injective stalk maps (by definition of the equivalence relation), the φ_x can be extended to $\tilde{\varphi}_x : \mathcal{Q}_x \rightarrow \mathcal{I}_x$ which defines $\tilde{\varphi} : \mathcal{Q} \rightarrow \mathcal{I}$. $\square$

As suggested in [5], we can use the above result to construct an injective resolution for any sheaf $\mathcal{M}$ on X . We know that $\mathcal{M}$ can be embedded into an injective sheaf $\mathcal{I}^0$ via σ . The cokernel of σ , $\text{coker } \sigma = \mathcal{I}^0 / \text{im } \sigma$. For any $U \subset X$ and $x \in \mathcal{I}^0(U)$, we can define the quotient map as $\pi(U)(x) = [x]$ where $[x]$ is the equivalence class given by the quotient map of $\mathcal{R}(U)$ -modules $\mathcal{I}^0 \rightarrow \mathcal{I}^0(U) / \text{im } \sigma(U)$. This is well-defined, and by definition of the restriction maps of the quotient sheaf, π is a morphism of sheaves. Now again by the previous result, we obtain an embedding $\sigma^* : \text{coker } (\sigma) \hookrightarrow \mathcal{I}^1$ into injective $\mathcal{I}^1$.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{M} & \xhookrightarrow{\sigma} & \mathcal{I}^0 & \xrightarrow{\pi \circ \sigma^*} & \mathcal{I}^1 \longrightarrow \dots \\
 & & & & \searrow \pi & & \swarrow \sigma^* \\
 & & & & & \text{coker } \sigma &
 \end{array}$$

Now clearly, $\ker \pi = \text{im } \sigma$ and since σ^* injective, $\ker \sigma^* \circ \pi = \ker \pi = \text{im } \sigma$. So by applying this step repeatedly we obtain an injective resolution of $\mathcal{M}, \mathcal{I}$. We can state this result as the following corollary.

Corollary 3.4.5. *Any sheaf $\mathcal{M}$ admits an injective resolution $\mathcal{I}$.*

We will use this property of injectivity repeatedly throughout this chapter. Our next application is what called the comparison theorem which states that any morphism of sheaves induces a morphism of cochain complexes on their respective injective resolutions.

3.4.1 The comparison theorem

The following theorem is an analogue of theorem 2.2.6 in [8] for injectives. This theorem will play a fundamental role in our definition of cohomology via resolution (that it is well-defined).

Theorem 3.4.6 (The comparison theorem). *Let A, B be sheaves on X and $\mathcal{I}, \mathcal{J}$ be corresponding injective resolutions. Then a morphism of sheaves $\varphi : A \rightarrow B$ extends to a map of complexes $\varphi : \mathcal{I} \rightarrow \mathcal{J}$. Moreover, this extension is unique up to homotopy.*

Proof. Proof by author. We label the maps as indicated in the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\varepsilon_A} & I^0 & \xrightarrow{d_I^0} & I^1 \longrightarrow \dots \\ & & \downarrow \varphi & & & & \\ 0 & \longrightarrow & B & \xrightarrow{\varepsilon_B} & J^0 & \xrightarrow{d_J^0} & J^1 \longrightarrow \dots \end{array}$$

Existence: We construct a cochain map inductively. The base case follows by the exactness of the first row, A maps via a sheaf map injectively into I^0 . Also A maps into J^0 via $\varepsilon_B \circ \varphi$, so by injectivity of J^0 , there exists a morphism $\varphi^0 : I^0 \rightarrow J^0$ such that $\varepsilon_B \circ \varphi = \varphi^0 \circ \varepsilon_A$.

Suppose $n = 1$, by the first isomorphism theorem for sheaves, $I^0/\text{im } \varepsilon_A \xrightarrow{\tilde{d}_I^0} I^1$. Also note that, for all $a \in \text{im } \varepsilon_A$, $d_J^0 \circ \varphi^0(a) = 0$ since by commutativity, $\varphi^0 \circ \varepsilon_A = \varepsilon_B \circ \varphi$ which maps into $\ker d_J^0$. Hence we can define $\tilde{d}_J^0 \circ \varphi^0$ on the quotient as,

$$\widetilde{d_J^0 \circ \varphi^0} : I^0/\text{im } \varepsilon_A \rightarrow J^1, [s] \mapsto [d_J^0 \circ \varphi^0(s)]$$

which is well-defined by the previous claim. Thus by injectivity of J^1 we can extend $\widetilde{d_J^0 \circ \varphi^0}$ via $\tilde{d}_I^0$ to $\varphi^1 : I^1 \rightarrow J^1$ with the property that $\varphi^1 \circ \tilde{d}_I^0 = \widetilde{d_J^0 \circ \varphi^0}$. However, on $\text{im } \varepsilon_A$, d_I^0 is zero so $\varphi^1 \circ d_I^0$ is zero, as is $d_J^0 \circ \varphi^0$ since $\text{im } \varphi^0 \circ \varepsilon_A \subset \ker d_J^0$ by an above argument. Using this we conclude that $\varphi^1 \circ d_I^0 = d_J^0 \circ \varphi^0$.

Now assume φ^i is defined for all $i \leq n$. By exactness of the first row, $\ker d_I^n = \text{im } d_I^{n-1}$, so by the first isomorphism theorem, $I^n/\text{im } d_I^{n-1} \xrightarrow{\tilde{d}_I^n} I^{n+1}$. By a similar argument as for the case $n = 1$ we can define a map $\widetilde{d_J^n \circ \varphi^n} : I^n/\text{im } d_I^{n-1} \rightarrow J^1$. By similar reasoning as for $n = 1$, injectivity of J^1 gives us a map $\varphi^{n+1} : I^{n+1} \rightarrow J^{n+1}$ such that $\varphi^{n+1} \circ d_I^n = d_J^n \circ \varphi^n$. Consequently, we obtain a cochain map $\varphi : \mathcal{I} \rightarrow \mathcal{J}$.

Uniqueness up to homotopy: Recall that the extending maps via some inclusion ι using injectivity of the codomain is not guaranteed to be unique. Hence the construction above is not generally unique. However we can prove it to be unique up to homotopy. For the given morphism $\varphi : A \rightarrow B$ consider two extensions $\varphi, \psi : \mathcal{I} \rightarrow \mathcal{J}$ depicted in the following commutative diagram,

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\varepsilon_A} & I^0 & \xrightarrow{d_I^0} & I^1 \longrightarrow \dots \\ & & \downarrow \varphi & & \downarrow \varphi^0 & \downarrow \psi^0 & \downarrow \varphi^1 & \downarrow \psi^1 \\ 0 & \longrightarrow & B & \xrightarrow{\varepsilon_B} & J^0 & \xrightarrow{d_J^0} & J^1 \longrightarrow \dots \end{array}$$

we claim these are homotopic. Hence we want to construct a sequence of maps $h^n : I^n \rightarrow J^{n-1}$ such that $\varphi^n - \psi^n = d_J^{n-1} \circ h^n + h^{n+1} \circ d_I^n$ for all strictly positive $n \in \mathbb{N}_{>0}$. We once again approach this via induction. For the base case let $n = 1$. Then for any $U \subset X$ open and $x \in I^0(U)$ define $h_{\text{im pre } d_I^0}^1(U)(d_I^0(U)(x)) = (\varphi^0 - \psi^0)(U)(x)$. So $h_{\text{im pre } d_I^0}^1 : \text{im pre } d_I^0 \rightarrow J^{n-1}$ is well-defined since $\varphi^0 = \psi^0$ on $\ker d_I^0$. Namely let $x \in \ker d_I^0(U)$ for some open $U \subset X$. Note that $\ker d_I^0(U) = (\text{im } \varepsilon_A)(U)$ and the image presheaf is already a

sheaf by the same reasoning as in the proof of i. in Lemma 3.2.4 since ε_A is injective (**monomorphism**). Now by commutativity,

$$(\varphi^0 - \psi^0) \circ \varepsilon_A = \varepsilon_B \circ (\varphi - \psi) = 0$$

implies $\varphi^0 = \psi^0$ on $\ker d_I^0$.

By sheafification, there exists a unique map $h_{\text{im } d_I^0}^1 : \text{im } d_I^0 \rightarrow J^0$ such that $h_{\text{im } d_I^0}^1 \circ \iota_{\text{im pre } d_I^0} = h_{\text{im pre } d_I^0}^1$. Now, since $\text{im } d_I^0$ is a subsheaf of I^1 there is an inclusion morphism $\iota : \text{im } d_I^0 \hookrightarrow I^1$. So by injectivity of J^0 , $h_{\text{im } d_I^0}^1$ can be extended to a morphism of sheaves $h^1 : I^1 \rightarrow J^0$ such that $h^1 \circ \iota = h_{\text{im } d_I^0}^1$ and

$$h^1 \circ d_I^0 = h_{\text{im } d_I^0}^1 \circ d_I^0 = h_{\text{im pre } d_I^0}^1 \circ d_I^0 = \varphi^0 - \psi^0.$$

Now suppose we have $k-1$ maps $h^n : I^n \rightarrow J^{n-1}$ such that $\varphi^n - \psi^n = d_J^{n-1} \circ h^n + h^{n+1} \circ d_I^n$ for all $0 < n < k-1$. By construction we want $h^k \circ d_I^{k-1} = \varphi^{k-1} - \psi^{k-1} - d_J^{k-2} \circ h^{k-1}$. This allows us to define h^k on the image of d_I^{k-1} which we denote $h_{\text{im pre } d_I^{k-1}}^k(U) := h^k(U)|_{\text{im } (d_I^{k-1}(U))}$. For this to be well-defined it must be zero on $\ker d_I^{k-1}(U)$.

Let $x \in \ker d_I^{k-1}(U) = \text{im } d_I^{k-2}(U)$. By Lemma 3.2.5, we know that as $\text{im } d_I^{k-2}$ is the sheafification of subpresheaf $\text{im pre } d_I^{k-2}$ of I^{k-1} , for all $y \in U$ there exists an open neighborhood $W_y \subset U$ such that $\rho_{W_y}^U(x) \in \text{im pre } d_I^{k-2}(W_y)$. So in other words there exists an $\tilde{x} \in I^{k-2}(W_y)$ such that $d_I^{k-2}(W_y)(\tilde{x}) = \rho_{W_y}^U(x)$. By commutativity of the cochain map with the complexes,

$$d_J^{k-2}(W_y)(\varphi^{k-2} - \psi^{k-2})(W_y)(\tilde{x}) = (\varphi^{k-1} - \psi^{k-1})(W_y) \circ d_I^{k-2}(W_y)(\tilde{x}).$$

The left hand side simplifies by the induction hypothesis to (where for $k=2$ the term $d_J^{k-3}(W_y) \circ h^{k-2}(W_y)(\tilde{x})$ needs to be omitted to obtain the equality from the base case),

$$d_J^{k-2}(W_y) (d_J^{k-3}(W_y) \circ h^{k-2}(W_y)(\tilde{x}) + h^{k-1}(W_y) \circ d_I^{k-2}(W_y)(\tilde{x})) = d_J^{k-2}(W_y) \circ h^{k-1}(W_y)(\rho_{W_y}^U(x))$$

and since both d_J^{k-2} and h^{k-1} are morphisms, it equals $\rho_{W_y}^U(d_J^{k-2}(U) \circ h^{k-1}(U)(x))$. The right hand side simplifies to $(\varphi^{k-1} - \psi^{k-1})(W_y) \circ (\rho_{W_y}^U(x)) = \rho_{W_y}^U((\varphi^{k-1} - \psi^{k-1})(U)(x))$. So the equality simplifies to,

$$\rho_{W_y}^U(d_J^{k-2}(U) \circ h^{k-1}(U)(x)) = \rho_{W_y}^U((\varphi^{k-1} - \psi^{k-1})(U)(x))$$

for all $y \in U$. So by locality of J^{k-1} , we obtain the desired equality $(\varphi^{k-1} - \psi^{k-1})(U)(x) = d_J^{k-2}(U) \circ h^{k-1}(U)(x)$.

Similarly as to the case $n=1$, we obtain by sheafification a unique map $h_{\text{im } d_I^{k-1}}^k$ such that $h_{\text{im } d_I^{k-1}}^k \circ \iota_{\text{im pre } d_I^{k-1}} = h_{\text{im pre } d_I^{k-1}}^k$. Since $\text{im } d_I^{k-1}$ is a subsheaf of I^k there is an inclusion map $\text{im } d_I^{k-1} \hookrightarrow I^k$ via which $h_{\text{im } d_I^{k-1}}^k$ extends to $h^k : I^k \rightarrow J^{k-1}$ by injectivity of J^{k-1} , such that $h^k \circ \iota = h_{\text{im } d_I^{k-1}}^k$. For our obtained h^k we have that,

$$h^k \circ d_I^{k-1} = h_{\text{im } d_I^{k-1}}^k \circ d_I^{k-1} = h_{\text{im pre } d_I^{k-1}}^k \circ d_I^{k-1} = \varphi^{k-1} - \psi^{k-1} - d_J^{k-2} \circ h^{k-1}$$

proving the induction step. Therefore, by induction φ and ψ are homotopic. $\square$

With this theorem in hand, we are ready to formally define cohomology via injective resolutions.

3.5 Cohomology via injectives

Recall that given any sheaf, we can obtain an injective resolution by Corollary 3.4.5, a consequence of the category of sheaves admitting enough injectives. As we will see in this section, this fact along with the comparison theorem allow us to define cohomology.

Definition 3.5.1 (Cohomology). Consider the evaluation functor $\Gamma_X : \mathbf{Sh}_{\mathcal{R}\text{-mod}}(X) \rightarrow \mathcal{R}(X)\text{-mod}$ from the category of sheaves over X to the category of $\mathcal{R}(X)$ -modules, such that $\mathcal{M} \mapsto \mathcal{M}(X)$. For any sheaf $\mathcal{M}$ with injective resolution $\mathcal{I}$, the **n -th cohomology** module is defined as,

$$H^n(X; \mathcal{M}) = \ker(d^n(X) : I^n(X) \rightarrow I^{n+1}(X)) / \text{im}(d^{n-1}(X) : I^{n-1}(X) \rightarrow I^n(X)).$$

There are $\mathcal{R}(X)$ -modules and $H^n(X; -) : \mathbf{Sh}_{\mathcal{R}\text{-mod}}(X) \rightarrow \mathcal{R}(X)\text{-mod}$ is in fact a functor (as we shall see in Theorem 4.1.1). $\diamond$

Do note that since the image is sheafified $\text{im}(d^n(X))$ is not necessarily equal to $\ker d^n(X)$. Hence due to Γ_X not preserving exactness, we say that Γ_X is not an exact functor. Thus the cohomologies can be non-trivial for $n > 0$. We will prove that cohomology does not depend on the choice of injective resolution, up to isomorphism. Before doing this, let us prove a first result. The following proposition and proof (which we modified slightly for conciseness) embody Proposition 4.4.1 of [5].

Proposition 3.5.2. [5] *Given sheaf $\mathcal{M}$ and injective sheaf $\mathcal{I}$ over X ,*

$$i. H^0(X; \mathcal{M}) \cong \mathcal{M}(X)$$

$$ii. H^n(X; \mathcal{I}) \cong 0 \text{ for all } n > 0$$

Proof. i. Take injective resolution $\mathcal{J}$, such that,

$$0 \longrightarrow \mathcal{M} \xrightarrow{\sigma} J^0 \xrightarrow{d^0} J^1 \xrightarrow{d^1} J^2 \longrightarrow \dots$$

then the 0-th cohomology is

$$H^0(X; \mathcal{M}) = \ker(J^0(X) \rightarrow J^1(X)) / \text{im}(0 \rightarrow I^0(X)) = \ker(I^0(X) \rightarrow I^1(X)).$$

Let $m \in \mathcal{M}(X)$. By exactness, $\sigma(X)(m) \in \ker d^0(X) = H^0(X; \mathcal{M})$. Since $\sigma(X)$ is injective, $\mathcal{M}(X)$ embeds into $H^0(X; \mathcal{M})$ via $\sigma(X)$. Now suppose $b \in H^0(X; \mathcal{M})$, as detailed in the proof of Lemma 3.2.4 i., $\text{im } \sigma$ is already a sheaf as presheaf, hence by exactness, $b \in \text{im}(\sigma)(X)$, so $\mathcal{M}(X) \cong \text{im}(\sigma)(X) = \ker d^0(X)$, the desired result.

$$ii. \text{ Since } \mathcal{I} \text{ injective, we can define injective resolution } I^n = \begin{cases} I & n = 0 \\ 0 & \text{else} \end{cases}. \text{ We can easily verify that}$$

$H^n(X; \mathcal{I}) = 0$ for any $n > 0$ under this resolution. $\square$

Now let us prove cohomology does not depend on the choice of injective resolution.

Lemma 3.5.3. *The n -th cohomology $H^n(X; \mathcal{M})$ is well-defined up to isomorphism of cochain complexes. That is, given two resolutions $\mathcal{I}$ and $\mathcal{J}$, the cohomology groups are isomorphic.*

Proof. By the Comparison theorem (Theorem 3.4.6), the identity map $\text{id}_{\mathcal{M}}$ induces a cochain map $f : \mathcal{I}^\bullet \rightarrow \mathcal{J}^\bullet$ that is unique up to homotopy,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M} & \longrightarrow & I^0 & \longrightarrow & I^1 \longrightarrow \dots \\ & & \text{id}_{\mathcal{M}} \downarrow & & f^0 \downarrow & & f^1 \downarrow \\ 0 & \longrightarrow & \mathcal{M} & \longrightarrow & J^0 & \longrightarrow & J^1 \longrightarrow \dots \end{array}$$

So for any other cochain map induced by the identity map on $\mathcal{M}$, f' , $f_* = f'_*$ by Lemma 3.3.8 ii.. By the same reasoning, we obtain a $g : \mathcal{I}^\bullet \rightarrow \mathcal{J}^\bullet$ that is unique up to homotopy. Note that $g \circ f$ and $\text{id}_{\mathcal{I}^\bullet}$ are both cochain maps that lift the identity $\text{id}_{\mathcal{M}}$. So they are homotopic and thus,

$$g_* \circ f_* = (g \circ f)_* = (\text{id}_{\mathcal{I}^\bullet})_* = \text{id}_{H_{\mathcal{I}^\bullet}^i(X; \mathcal{M})}$$

where the first equality follows from the definition of the pushforward of cochain maps, and the final map is the identity on the i -th cohomology module resolved via resolution $\mathcal{I}$. Similarly $f_* \circ g_* = \text{id}_{H_{\mathcal{J}^\bullet}^i(X; \mathcal{M})}$. Therefore f_* is an isomorphism of cochain complexes, proving that $H^n(X; \mathcal{M})$ is well-defined. $\square$

4 Computing Cohomology

In practice injective resolutions may not be the most convenient approach via which to resolve the cohomology modules. In this chapter we will discuss other approaches used, in particular we will show that a larger class of resolutions induce isomorphic cohomologies. Before discussing the class of acyclic resolutions with this property, we first develop some useful results pertaining to the sheaf cohomology. While these results will not be used extensively in this thesis, the results are nonetheless included due to their importance.

4.1 Functoriality of $H^n(-, -)$

In this section we discuss the functoriality of $H^n(-, -)$, that is how $H^n(-, -)$ interacts with continuous maps between topological spaces and sheaf morphisms. Although these relationships are not used extensively in this thesis, it is worth mentioning them. Especially, since the functionality in the second variable is important in establishing the homotopy invariance of sheaf cohomology, confirming that sheaf cohomology is an invariant. The functorial relationship is described by the following theorem (Proposition 4.5.1 in [5]),

Theorem 4.1.1 (Functoriality). *Given sheaves A, B on X .*

- i. *A morphism of sheaves $\varphi : A \rightarrow B$ induces an $\mathcal{R}(X)$ -module homomorphism $\varphi_* : H^\bullet(X; A) \rightarrow H^\bullet(X; B)$.*
- ii. *A continuous map $f : Y \rightarrow X$ induces an $\mathcal{R}$ -module homomorphism $f_* : H^\bullet(X; A) \rightarrow H^\bullet(Y; f^*A)$.*

Proof. The proof is as in [5] with some additions by the author.

i. By the comparison theorem, Theorem 3.4.6, φ induces a cochain map φ_* between injective resolution of $A, \mathcal{I}$, and an injective resolution of B , and $\mathcal{J}$ that is unique up to homotopy. Now recall in Lemma 3.3.2, we proved that this cochain map can be pushed forward to a homomorphism of cohomologies φ_* .

ii. Resolve f^*A by an injective resolution $\mathcal{J}$. Applying f_* to this sequence, we see that each f_*J^n is injective. Namely by the adjunction formula any function $\varphi : B \rightarrow f^*I^n$ corresponds to some $\psi : f_*B \rightarrow I^n$. Let $B \xrightarrow{\iota} C$, then $f_*B \xrightarrow{f_*\iota} f_*C$, so by injectivity of I^n , ψ extends via $f_*\iota$ to some $\tilde{\psi}$, which corresponds via the injection formula to some $\tilde{\varphi} : C \rightarrow f^*I^n$, since $\tilde{\psi}|_{f_*B} = \tilde{\psi} \circ f_*\iota = \psi$, corresponds to φ , $\tilde{\varphi}|_B = \tilde{\varphi} \circ \iota = \varphi$.

By the adjunction formula, from id_{f^*A} , we obtain a map $\eta_A : A \rightarrow f_*f^*A$. Resolving A by a resolution $\mathcal{I}$, we obtain a morphism, $\eta_A : \mathcal{I} \rightarrow f_*\mathcal{J}$ by the comparison theorem. Now by lemma 3.3.2, this gives us a homomorphism $H^n(X; A) \rightarrow H^n(X; f_*f^*A)$.

The final part of the proof involves showing that $H^\bullet(X; f_*f^*A) \cong H^\bullet(Y; f^*A)$. To show this we define a class of maps $\eta_S : S(Y) \rightarrow f_*(S)(X)$ given sheaves S over Y . Firstly, for any sheaf S , note that $f_*(S)(X) = S(f^{-1}(X)) = S(Y)$ so we define each of these η_S by the identity, hence these are each isomorphisms of sheaves. Secondly, given any morphism $g : S \rightarrow T$, we claim that

$$\eta_T \circ \Gamma_Y(g) = \Gamma_X \circ f_*(g) \circ \eta_S.$$

Let $s \in S(Y)$. Then,

$$\begin{aligned} \Gamma_X \circ f_*(g) \circ \eta_S(s) &= \Gamma_X \circ f_*(g)(s) \\ &= g(f^{-1}(X))(s) = g(Y)(s) \end{aligned}$$

Also obviously $\eta_T(g(Y)(s)) = g(Y)(s)$, proving the claim.

Next resolve f^*A by injectives $\mathcal{I}$. Then, $\Gamma_Y(I^i) = \Gamma_X \circ f_*(I^i)$ for all i , by the relation we established above. Since $\eta_{I^{n+1}} \circ \Gamma_Y(d^n) = \Gamma_X \circ f_*(d^n) \circ \eta_{I^n}$ and $\Gamma_Y(d^{n+1}) \circ \Gamma_Y(d^n) = 0 = \Gamma_X \circ f_*(d^{n+1}) \circ \Gamma_X \circ f_*(d^n)$, the map

$$\eta_{I^\cdot} : \Gamma_Y(\mathcal{I}^\cdot) \rightarrow \Gamma_X \circ f_*(\mathcal{I}^\cdot)$$

is a cochain map. Since each of the η_{I^n} are the identity, this induces an isomorphism on cohomology. Thus $H^\cdot(Y; f^*A) = H^\cdot(\Gamma_Y(\mathcal{I}^\cdot)) \cong H^\cdot(\Gamma_X \circ f_*(\mathcal{I}^\cdot)) = H^\cdot(X; f_*f^*A)$, completing the proof. $\square$

4.1.1 Homotopy Invariance of Sheaf Cohomology

In this section we will briefly mention the homotopy invariance of sheaf cohomology on locally constant sheaves. The proof of this property is beyond the scope of this thesis and uses concepts such as proper base change, soft sheaves and cohomology with compact supports. The main theorem of this section is Theorem 8.2.1 [5], and is adapted to the setting of sheaves over abelian groups. First we define homotopy.

Definition 4.1.2. Let X, Y be topological spaces. We say that two continuous functions $f, g : X \rightarrow Y$ are **homotopic** if there exists a continuous function $F : X \times [0, 1] \rightarrow Y$ such that, $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. $\diamond$

Recall also that locally constant sheaves on X , A are sheaves with the property that for all $x \in X$ there exists a neighborhood U_x such that the restricted subsheaf $A|_{U_x}$ is isomorphic to some constant sheaf $\underline{A}_x$.

Theorem 4.1.3 (Theorem 8.2.1 [5]). *Let X, Y be topological spaces, A a locally constant sheaf of abelian groups on X and $f, g : Y \rightarrow X$ be homotopic. Then, f and g induce the cochain maps on cohomology. That is, by the functoriality, f, g induce cochain maps $f^\cdot : H^\cdot(X; A) \rightarrow H^\cdot(Y; f^*A)$ and $g^\cdot : H^\cdot(X; A) \rightarrow H^\cdot(Y; f^*A)$ such that $f^\cdot = g^\cdot$.*

Proof. For the proof see [5]. $\square$

Definition 4.1.4. We say two topological spaces X, Y are **homotopy equivalent**, if there exist continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X . $\diamond$

Corollary 4.1.5. *Let X and Y be homotopy equivalent and A a locally constant sheaf of abelian groups over X then $H^\cdot(X; A) \cong H^\cdot(Y; f^*A)$.*

Proof. Well since X and Y are homotopy equivalent there exist maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that, $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X . By the previous theorem, $(f \circ g)^\cdot = \text{id}_Y^\cdot$ and $(g \circ f)^\cdot = \text{id}_X^\cdot$. Hence f and g are each others inverses and thus $H^\cdot(X; A) \cong H^\cdot(Y; f^*A)$. $\square$

4.2 Long Exact Sequence in Cohomology

The goal of this section is to prove that given a short exact sequence of sheaves, we can obtain a long exact sequence of their cohomologies. The following theorem is proposition 4.7.1 in [5]. We adapt proofs presented in some lecture notes by Robert Ash [9], Moerdijk and Pronk [5] and in a textbook by Weibel [8] to our setting.

Theorem 4.2.1 (Long exact sequence in cohomology). *Given an exact sequence of sheaves $0 \rightarrow A \xrightarrow{\psi} B \xrightarrow{\varphi} C \rightarrow 0$. We can construct a long exact sequence in cohomology,*

$$\begin{aligned} 0 \rightarrow H^0(X; A) \rightarrow H^0(X; B) \rightarrow H^0(X; C) \xrightarrow{\partial} H^1(X; A) \rightarrow \\ \rightarrow H^1(X; B) \rightarrow H^1(X; C) \rightarrow \cdots \rightarrow H^n(X; C) \xrightarrow{\partial} H^{n+1}(X; A) \rightarrow \cdots \end{aligned}$$

4.2.1 Two Results in Homological Algebra

In this subsection we introduce two useful lemmata from homological algebra that play an important role in our proof of the theorem.

Lemma 4.2.2 (Snake Lemma). *Consider the commutative diagram of R -modules,*

$$\begin{array}{ccccccc} & & A' & \xrightarrow{\psi'} & B' & \xrightarrow{\varphi'} & C' \longrightarrow 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h \\ 0 & \longrightarrow & A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C \end{array}$$

If the rows are exact then,

$$\ker f \rightarrow \ker g \rightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \rightarrow \operatorname{coker} g \rightarrow \operatorname{coker} h$$

is exact. With $\delta(c') = \psi^{-1} \circ g \circ \varphi'^{-1}(c')$ for $c' \in \ker h$

Proof. We follow the proof in [9]. First we can clearly extend the diagram to,

$$\begin{array}{ccccccc} \ker f & \xrightarrow{\tilde{\psi}'} & \ker g & \xrightarrow{\tilde{\varphi}'} & \ker h & & \\ \downarrow \iota_{\ker f} & & \downarrow \iota_{\ker g} & & \downarrow \iota_{\ker h} & & \\ A' & \xrightarrow{\psi'} & B' & \xrightarrow{\varphi'} & C' & \longrightarrow & 0 \\ \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C \\ \downarrow \pi_{\operatorname{coker} f} & & \downarrow \pi_{\operatorname{coker} g} & & \downarrow \pi_{\operatorname{coker} h} & & \\ \operatorname{coker} f & \longrightarrow & \operatorname{coker} g & \longrightarrow & \operatorname{coker} h & & \end{array}$$

The sequence is exact in $\ker g$ and $\operatorname{coker} g$:

We will prove the exactness in $\ker g$. We chase the diagram. By commutativity, $\varphi' \circ \psi' \circ \iota_{\ker f} = \iota_{\ker h} \circ \tilde{\varphi}' \circ \tilde{\psi}'(x) = 0$. Since $\iota_{\ker h}$ is injective, $\tilde{\varphi}' \circ \tilde{\psi}' = 0$, so $\operatorname{im} \tilde{\psi}' \subset \ker \tilde{\varphi}'$.

Now let $x \in \ker \tilde{\varphi}'$. Then $\varphi' \circ \iota_{\ker g}(x) = \iota_{\ker h} \circ \tilde{\varphi}'(x) = 0$ so $\iota_{\ker g}(x) \in \ker \varphi' = \operatorname{im} \psi'$. So there exists a $y \in A'$ such that $\psi'(y) = \iota_{\ker g}(x)$, and $g \circ \psi'(y) = \psi \circ f(y) = 0$ so since ψ injective, $y \in \ker f$. So $\iota_{\ker g}(x) = \psi' \circ \iota_{\ker f}(y) = \iota_{\ker g} \circ \tilde{\psi}'(y)$ implies that $\tilde{\psi}'(y) = x$ so $x \in \operatorname{im} \tilde{\psi}'$. By another diagram chase now using the surjectivity of φ' , we conclude the sequence is exact in $\operatorname{coker} g$.

The map δ does not depend on choice of representatives

Let $c' \in \ker h$ and $b, b' \in B'$ such that $\varphi'(b) = \varphi'(b') = c'$. Then $b - b' \in \ker \varphi' = \operatorname{im} \psi'$ so there exists an $a \in A'$ such that $\psi'(a) = b - b'$. By commutativity, $\psi \circ f(a) = g \circ \psi'(a) = g(b - b')$.

Note also that $h \circ \varphi'(b) = \varphi \circ g(b) = 0$ so $g(b) \in \ker \varphi = \operatorname{im} \psi$. So there exists a $\tilde{b} \in A$ such that $\psi(\tilde{b}) = g(b)$. We can similarly find a $\tilde{b}'$ for b' . Now $\psi(\tilde{b} - \tilde{b}') = g(b - b') = \psi \circ f(a)$, so by injectivity of ψ , $\tilde{b} - \tilde{b}' = f(a)$ hence $[\tilde{b}] = [\tilde{b}']$ in $\operatorname{coker} f$. So $\delta(c')$ does not depend on the choice of the b and $\tilde{b}$.

δ is an R -module homomorphism

Consider $c', d' \in \ker h$ and $r \in R$, we prove that $\delta(c' + rd') = \delta(c') + r\delta(d')$. As above, choose c, d such that $\varphi'(c) = c'$ and $\varphi'(d) = d'$ so $\varphi'(c + rd) = c' + rd'$. Now $g(c + rd) = g(c) + rg(d)$. Now again, choose $\tilde{c}, \tilde{d} \in A$ as above such that again $\varphi(\tilde{c}) = g(c)$ and $\varphi(\tilde{d}) = g(d)$. Then $\psi(\tilde{c} + r\tilde{d}) = g(c + rd)$. By definition of the quotient module $[\tilde{c} + r\tilde{d}] = [\tilde{c}] + r[\tilde{d}]$, this gives us the desired result.

The sequence is exact

To prove the exactness we need to show exactness in $\ker h$ and $\operatorname{coker} f$. We do the diagram chase for $\ker h$, noting that exactness in $\operatorname{coker} f$ follows by an analogous argument.

Let $z \in \operatorname{im} \tilde{\varphi}'$, the preimage y is in $\ker g$ so $\partial(z) = 0$. Now let $z \in \ker \delta$. Choose $z' \in B'$ and $\tilde{z} \in A$ defined similarly to b and $\tilde{b}$ in the independence of choice claim, so that $g(z') = \psi(\tilde{z})$. Hence $[\tilde{z}] = \delta(z') = 0$ so $\tilde{z} \in \operatorname{im} f$, thus there exists an $a \in A'$ such that $f(a) = \tilde{z}$. So $g(z') = \psi(\tilde{z}) = \psi \circ f(a) = g \circ \psi'(a)$ and consequently, $z' - \psi'(a) \in \ker g$. However $z = \varphi'(z')$ so $z = \varphi'(z') = \varphi'(z' - \psi'(a) + \psi'(a)) = \varphi'(z' - \psi'(a)) \in \operatorname{im} \tilde{\varphi}'$. $\square$

The snake lemma will be used in the proof of the version of Theorem 4.2.1 for cochain complexes of modules. Although we could this prove version directly via a diagram chase, we decided to include this lemma due to its importance in the homological algebra [8].

In the construction of the long exact sequence of cohomology we require a short exact sequence of cochain complexes. Since we only begin with a short exact sequence of sheaves, we need the following result, which states that given a short exact sequence of sheaves, we can find resolutions of the elements of this sequence that form a short exact sequence.

Lemma 4.2.3 (Horseshoe Lemma for Injectives). *Given a short exact sequence of sheaves, $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, we can resolve each sheaf via injectives $A \rightarrow \mathcal{I}$, $B \rightarrow \mathcal{J}$ and $C \rightarrow \mathcal{K}$ such that $0 \rightarrow \mathcal{I} \rightarrow \mathcal{J} \rightarrow \mathcal{K} \rightarrow 0$ is a short exact sequence of resolutions.*

Proof. The following proof is written by the author, with inspiration from the proof of the statement for projectives (the dual concept to injectives) in [8]. First we prove the following claim,

Claim 1. Given an exact diagram of sheaves,

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 A & \xrightarrow{\iota_A} & A' \\
 \downarrow \psi & & \downarrow \psi' \\
 B & \xrightarrow{\sigma} & A' \oplus C' \\
 \downarrow \varphi & & \downarrow \varphi' \\
 C & \xrightarrow{\iota_C} & C' \\
 & & \downarrow \\
 & & 0
 \end{array}$$

where A', C' are injectives and $\psi'(a') = (a', 0)$, $\varphi'(a', c') = c'$. There exists a monomorphic morphism of sheaves $\sigma : B \rightarrow A' \oplus C'$ such that the diagram commutes.

We begin by defining σ . Since A' injective, we can extend ι_A via ψ to $\tilde{\iota}_A : B \rightarrow A'$. Now define $\sigma := (\tilde{\iota}_A, \iota_C \circ \varphi) : B \rightarrow A' \oplus C'$. Clearly, $\sigma \circ \psi = (\tilde{\iota}_A \circ \psi, \iota_C \circ \varphi \circ \psi) = (\iota_A, 0) = \psi' \circ \iota_A$ since the first column is exact and by definition of the extension via injectivity, also by definition of φ' , $\varphi' \circ \sigma = \iota_C \circ \varphi$.

Now we prove $\ker \sigma = 0$. This follows by a simple diagram chase, namely let $U \subset X$ and $x \in \ker \sigma(U)$. Then by commutativity, $\iota_C(U) \circ \varphi(U)(x) = \varphi'(U) \circ \sigma(U)(x) = 0$ so by injectivity of $\iota_C(U)$, $x \in \ker \varphi(U) = \text{im } \psi(U)$ by exactness of the first column. Since ψ is a monomorphism, we find by Lemma 3.2.6, that there exists a $a \in A(U)$ such that $\psi(U)(a) = x$. Now since $\psi'(U) \circ \iota_A(U)(a) = \sigma(U) \circ \psi(U)(a) = 0$, $\psi'(U)$ maps $\iota_A(U)(a)$ to $(\iota_A(U)(a), 0) = 0$. So $\iota_A(U)(a) = 0$, and thus by injectivity of $\iota_A(U)$, $a = 0$. Therefore, $x = 0$ and hence $\ker \sigma(U) = 0$, proving the claim.

Now we can prove the result inductively using the claim. The base case following immediately, where we define $J^0 := I^0 \oplus K^0$. By projecting the right column into the cokernels, we can obtain the diagram,

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
A & \xleftarrow{\varepsilon_A} & I^0 & \longrightarrow & \text{coker } \varepsilon_A \\
\downarrow \psi & & \downarrow \psi' & & \downarrow \\
B & \xleftarrow[\varepsilon_B]{} & I^0 \oplus K^0 & \longrightarrow & \text{coker } \varepsilon_B \\
\downarrow \varphi & & \downarrow \varphi' & & \downarrow \\
C & \xleftarrow{\varepsilon_C} & K^0 & \longrightarrow & \text{coker } \varepsilon_C \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

Claim 2. The cokernel column is exact. This follows by an elicited diagram chase.

Of note is $\text{coker } \varepsilon_A \xrightarrow{\tilde{\psi}'} \text{coker } \varepsilon_B$. Let $U \subset X$ open, $[x] \in \text{coker } \varepsilon_A(U)$. Note that $\tilde{\psi}'(U)$ is injective if, and only if, $\psi'(U)(x) \in \text{im } \varepsilon_B(U)$ implies that $x \in \text{im } \varepsilon_A(U)$. Note that since ε_B is a monomorphism, there exists an $\tilde{x} \in B(U)$ such that $\varepsilon_B(U)(\tilde{x}) = \psi'(U)(x)$. Now $0 = \varphi'(U) \circ \psi'(U)(x) = \varphi'(U) \circ \varepsilon_B(U)(\tilde{x})$ which by commutativity equals $\varepsilon_C(U) \circ \varphi(U)(\tilde{x})$. So since ε_C is a monomorphism, $\tilde{x} \in \ker \varphi(U) = \text{im } \psi(U)$. Now since ψ is a monomorphism, there exists a $y \in A(U)$ such that $\psi(U)(y) = \tilde{x}$. Therefore, by commutativity, $\psi'(U)(x) = \varepsilon_B(U) \circ \psi(U)(y) = \psi'(U) \circ \varepsilon_A(U)(y)$. So since $\psi'(U)$ is injective, $x = \varepsilon_A(U)(y)$ proving $\tilde{\psi}'(U)$ is injective. By similar diagram chases we conclude the exactness of the third column.

Using the construction of the injective resolutions, we find inclusions $d' : \text{coker } \varepsilon_A \hookrightarrow I^1$ and $\tilde{d} : \text{coker } \varepsilon_C \hookrightarrow K^1$. Applying the claim we obtain a map $d : \text{coker } \varepsilon_B \hookrightarrow I^1 \oplus K^1$ where we define $J^1 := I^1 \oplus K^1$. We can define the compositions of these maps with the projections as, d_J^1, d_K^1 and d_I^1 respectively. By construction $d_I^1 \circ \varepsilon_A = 0$, and the same holds for the d_K^1 . Since $\text{im } \varepsilon_B$ gets mapped to 0 under the projection onto the cokernel, also $d_J^1 \circ \varepsilon_B = 0$ and this sequence is also exact by the fact $\tilde{d}$ is monomorphic. We can repeat this step once again, where in each step the exactness of the short exact sequence of cokernels follows by claim 2, and so indefinitely to obtain our desired resolution $\mathcal{J}$ of B proving the claim. $\square$

Using these two lemmata we are now ready to prove the main result of this section.

4.2.2 Proof of Theorem 4.2.1

In this section we prove the theorem. We split the theorem up into several claims.

Proof. First note that from the short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of sheaves we can obtain the exact sequence of resolutions $0 \rightarrow \mathcal{I} \rightarrow \mathcal{J} \rightarrow \mathcal{K} \rightarrow 0$ using the Horseshoe lemma. Now using the second claim (which relies on the first claim) we find that $0 \rightarrow \Gamma_X \mathcal{I} \rightarrow \Gamma_X \mathcal{J} \rightarrow \Gamma_X \mathcal{K} \rightarrow 0$ is an exact sequence of cochain complexes. Now applying the third claim we obtain a long exact sequence in cohomology. Recall $H^n(X; A)$ was defined using the evaluation of the resolutions by the global sections functor Γ_X , so this is indeed our sought after long exact sequence.

Claim 1. For an injective sheaf A , the restrictions ρ_V^U are surjective.

This is the proof of lemma 4.7.1 in [5] adapted to our situation. Let $U \subset X$ be open, and define the sheaf $\mathcal{R}_U$ as the sheaf of locally constant functions with support $\overline{U}$, that is for any open $W \subset X$,

$$\mathcal{R}_U(W) = \{f : W \rightarrow A : f \text{ locally constant and } f(x) = 0 \forall x \notin U\}.$$

When $V \subset U$, there is an obvious inclusion $\mathcal{R}_V \hookrightarrow \mathcal{R}_U$.

We claim the morphisms $\varphi : \mathcal{R}_U \rightarrow A$ are in bijective correspondence with elements $a \in A(U)$. For a given φ , we define $a := \varphi(U)(1_U)$ (1_U being the constant function 1 on open $U \subset X$). In the other direction, for some $a \in A(U)$, for each $W \subset X$ and $f \in \mathcal{R}_U(W)$, define

$$\varphi(W)(f) = a \cdot f$$

where a acts on f via the module structure (using the fact f is locally constant of course; so locally on opens W_i).

Using the bijection any $a \in A(V)$ corresponds to an $f : \mathcal{R}_V \rightarrow A$. By the injectivity of A we can extend f via the inclusion $\iota : \mathcal{R}_V \hookrightarrow \mathcal{R}_U$, to a $\tilde{f} : \mathcal{R}_U \rightarrow A$ such that $\tilde{f} \circ \iota = f$. Now $\tilde{f}$ corresponds to some $\tilde{a} \in A(U)$ such that $\rho_V^U(\tilde{a})$ corresponds to $\tilde{f} \circ \iota = f$, so since the correspondence is bijective, $\rho_V^U(\tilde{a}) = a$.

Claim 2. If A is injective in the exact sequence of sheaves $0 \rightarrow A \xrightarrow{\psi} B \xrightarrow{\varphi} C \rightarrow 0$, then $0 \rightarrow A(X) \rightarrow B(X) \rightarrow C(X) \rightarrow 0$ is exact.

For the proof of this claim we use the proof of Lemma 4.7.3 in [5]. The exactness in $A(X)$ is clear since ψ is a monomorphism and the exactness in $B(X)$ follows by similar reasoning as in 3.5.2. So we are left to show that $B(X) \rightarrow C(X)$ is surjective. Let $c \in C(X)$. Consider the pairs (U, b) where $U \subset X$ and $b \in B(U)$ such that $\psi(U)(b) = \rho_U^X(c)$ and the partial order defined as $(U, b) \leq (U', b')$ if, and only if, $U \subset U'$ and $b = \rho_U^{U'}(b')$. By Zorn's lemma as on page 909 in [3], there exists a maximal pair (U, b) . Now assume that $U \neq X$, and let $x \in X \setminus U$, then by Lemma 3.2.5 there exists a neighborhood $U_x \subset U$ of x and a $b_x \in B(U_x)$ such that $\varphi(U_x)(b_x) = \rho_{U_x}^X(c)$. So

$$\varphi(U \cap U_x)(\rho_{U_x \cap U}^{U_x}(b_x)) = \varphi(U \cap U_x)(\rho_{U \cap U_x}^U(b)).$$

By similar reasoning as before, the sequence,

$$0 \rightarrow A(U_x \cap U) \rightarrow B(U_x \cap U) \rightarrow C(U_x \cap U) \rightarrow 0$$

is exact at $A(U_x \cap U)$ and $B(U_x \cap U)$. Thus since $\rho_{U_x \cap U}^{U_x}(b_x) - \rho_{U \cap U_x}^U(b) \in \ker \varphi(U \cap U_x) = \text{im } \psi(U \cap U_x)$

by Lemma 3.2.6, there exists an $a \in A(U_x \cap U)$ such that this equals $\psi(U \cap U_x)(a)$. By the previous claim, this $a \in A(U_x \cap U)$ can be extended to some $\tilde{a} \in A(X)$. However $b - \rho_U^X(\psi(X)(\tilde{a}))$ and b_x are equal on the restriction to $U_x \cap U$, so we can glue to obtain a $b' \in B(U_x \cup U)$ such that $\varphi(U_x \cup U)(b') = \rho_{U_x \cup U}^X(c)$ contradicting the maximality of U .

Claim 3. Given an exact sequence of cochain complexes $0 \rightarrow \mathcal{A}^\cdot \rightarrow \mathcal{B}^\cdot \rightarrow \mathcal{C}^\cdot \rightarrow 0$ of modules we can construct a long exact sequence in cohomology,

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{A}^\cdot) \rightarrow H^0(\mathcal{B}^\cdot) \rightarrow H^0(\mathcal{C}^\cdot) \xrightarrow{\partial} H^1(\mathcal{A}^\cdot) \rightarrow \\ \rightarrow H^1(\mathcal{B}^\cdot) \rightarrow H^1(\mathcal{C}^\cdot) \rightarrow \cdots \rightarrow H^n(\mathcal{C}^\cdot) \xrightarrow{\partial} H^{n+1}(\mathcal{A}^\cdot) \rightarrow \cdots \end{aligned}$$

The claim follows by 2 applications of the snake lemma, the following proof is standard and as in [8], [9].

To obtain our desired ∂ we will apply the snake lemma to,

$$\begin{array}{ccccccc} I^n/B^n(\mathcal{I}^\cdot) & \longrightarrow & J^n/B^n(\mathcal{J}^\cdot) & \longrightarrow & K^n/B^n(\mathcal{K}^\cdot) & \longrightarrow & 0 \\ \downarrow d_I & & \downarrow d_J & & \downarrow d_K & & \\ 0 & \longrightarrow & Z^{n+1}(\mathcal{I}^\cdot) & \longrightarrow & Z^{n+1}(\mathcal{J}^\cdot) & \longrightarrow & Z^{n+1}(\mathcal{K}^\cdot) \end{array}$$

Where we define the vertical maps as $d_I(x^n + B^n(\mathcal{I}^\cdot)) = d_I^n(x^n)$ and similarly for J and K . Now

$$\ker d_I = \ker d_I^n/B^n(\mathcal{I}^\cdot) = H^n(\mathcal{I}^\cdot), \quad \text{coker } d_I = Z^{n+1}(\mathcal{I}^\cdot)/\text{im } d_I^n = H^{n+1}(\mathcal{I}^\cdot).$$

To be able to apply the snake lemma, we must show that the rows are exact. For this, we apply the snake lemma to,

$$\begin{array}{ccccccc} 0 & \longrightarrow & I^n & \longrightarrow & J^n & \longrightarrow & K^n \longrightarrow 0 \\ & & \downarrow d_I^n & & \downarrow d_J^n & & \downarrow d_K^n \\ 0 & \longrightarrow & I^{n+1} & \longrightarrow & J^{n+1} & \longrightarrow & K^{n+1} \longrightarrow 0 \end{array}$$

To obtain,

$$Z^n \mathcal{I}^\cdot \rightarrow Z^n(\mathcal{J}^\cdot) \rightarrow Z^n(\mathcal{K}^\cdot) \xrightarrow{\delta} I^{n+1}/B^n(\mathcal{I}^\cdot) \rightarrow J^{n+1}/B^{n+1}(\mathcal{J}^\cdot) \rightarrow K^{n+1}/B^{n+1}(\mathcal{K}^\cdot).$$

In the case $n = 0$ we only need the exactness of the $J^0/B^0(\mathcal{J}^\cdot)$ term which also follows by the first claim of the snake lemma. We can easily prove that since $I^{n+1} \hookrightarrow J^{n+1}$, $Z^{n+1}(\mathcal{I}^\cdot) \hookrightarrow Z^{n+1}(\mathcal{J}^\cdot)$, and since $J^n \rightarrow K^n$ is surjective, $J^n/B^n(\mathcal{J}^\cdot) \rightarrow K^n/B^n(\mathcal{K}^\cdot)$ is surjective, proving the desired exactness of rows. Now applying the snake lemma on the first diagram, we obtain module homomorphisms $\partial : H^n(\mathcal{K}^\cdot) \rightarrow H^{n+1}(\mathcal{I}^\cdot)$, such that

$$H^n(\mathcal{I}^\cdot) \rightarrow H^n(\mathcal{J}^\cdot) \rightarrow H^n(\mathcal{K}^\cdot) \xrightarrow{\partial} H^{n+1}(\mathcal{I}^\cdot) \rightarrow H^{n+1}(\mathcal{J}^\cdot) \rightarrow H^{n+1}(\mathcal{K}^\cdot)$$

is exact for all $n \in \mathbb{Z}_{\geq 0}$. Concatenating these sequences in the appropriate manner, and noting that by the argument in Proposition 3.5.2(i) (since we don't use injectivity of the resolution), $H^0(\mathcal{I}^\cdot) \cong A$, $H^0(\mathcal{J}^\cdot) \cong B$ thus $H^0(\mathcal{I}^\cdot) \hookrightarrow H^0(\mathcal{J}^\cdot)$ proving the claim final claim, completing the proof.

□

This result acts as an important tool in computing cohomology modules, since we can use the exactness

of the obtained long exact sequence to make some conclusions about said modules. For example, given a short exact sequence $0 \rightarrow A \xrightarrow{\varphi} B \rightarrow 0$, we know φ is an isomorphism. We now move onto the main topic of this chapter, comparing resolutions of sheaves. This will be important since resolving a sheaf via injectives is not necessarily straightforward. Hence we will want to resolve sheaves via some other resolutions that induce isomorphic cohomologies.

4.3 Comparing Resolutions

In this section we will go about comparing different resolutions. The proofs in the section are by the author with inspiration from the supervisor Marius Crainic and the stacks projects [10]. Given a sheaf A on X and an arbitrary resolution (exact sequence of sheaves), $0 \rightarrow A \rightarrow \mathcal{R}^\cdot$, we want to develop a means of comparing the induced cohomologies $H^n(\mathcal{R}^\cdot)$, to the standard cohomology modules $H^n(X; A)$ resolved by some injective resolution $\mathcal{I}^\cdot$. We will do this by a diagram chase of a complex of cochain complexes, before defining this we will prove that we can obtain a long exact sequence of injective resolutions from a long exact sequence, we will call this the Cartan-Eilenberg resolution. The following theorem (similar to Lemma 13.21.2 in [10]) is the long exact analogue of the Horseshoe lemma,

Lemma 4.3.1 (Cartan-Eilenberg Resolution; [10]). *Given a long exact sequence $0 \rightarrow A \rightarrow \mathcal{R}^\cdot$ of sheaves, we can resolve A and each R^i by injective resolutions $\mathcal{J}^n$ such that,*

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A & \longrightarrow & R^0 & \longrightarrow & R^1 \longrightarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & J^{0,0} & \longrightarrow & J^{0,1} & \longrightarrow & J^{0,2} \longrightarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & J^{1,0} & \longrightarrow & J^{1,1} & \longrightarrow & J^{1,2} \longrightarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

is a long exact sequence of resolutions.

Proof. The following proof is written by the author with inspiration from [10]. The proof involves nothing special. Note that for all $n \in \mathbb{Z}_{\geq 0}$, the sequence

$$0 \rightarrow \ker d_R^n \rightarrow R^n \rightarrow \operatorname{im} d_R^n \rightarrow 0$$

is exact. We resolve each $\ker d_R^n$ by some resolution $\mathcal{K}^n$, note that since the sequence is exact, $\operatorname{im} d_R^n = \ker d_R^{n+1}$ so we can resolve each $\operatorname{im} d_R^n$ also by $\mathcal{K}^{n+1}$. Now by the horseshoe lemma, we have a resolution $\mathcal{J}^{\cdot, n}$ of R^n such that $J^{i, n} = K^{i, n} \oplus K^{i, n+1}$. Where for all $U \subset X$, the maps $f^{i, n}(U) : J^{i, n}(U) \rightarrow K^{i, n+1}(U)$, $(x, y) \mapsto y$ are surjective for all i, n and the maps $g^{i, n}(U) : K^{i, n}(U) \rightarrow J^{i, n}(U)$, $x \mapsto (x, 0)$ are injective for all i, n . Now define $d_J^{i, n} := g^{i, n+1} \circ f^{i, n} : J^{i, n} \rightarrow J^{i, n+1}$, so that $d_J^{i, n}(U)(x, y) = (y, 0)$. Clearly, $\ker d_J^{i, n} = K^{i, n} \oplus 0$ and $\operatorname{im} d_J^{i, n} = K^{i, n+1} \oplus 0$, so $\ker d_J^{i, n+1} = \operatorname{im} d_J^{i, n}$. So the sequences $\mathcal{J}^{i, \cdot}$ are exact. Finally, for $\varepsilon_A : A \hookrightarrow R^0$ we have by the first isomorphism theorem for sheaves that $A \cong \operatorname{im} \varepsilon_A = \ker d_R^0$.

Hence by the comparison theorem, any resolution of A is isomorphic to $\mathcal{K}^{\cdot,0}$, which map into the $J^{i,0}$ via the monomorphisms $g^{i,0}$. By relabeling $\mathcal{J}^{\cdot,\cdot}$ including $\mathcal{K}^{\cdot,0}$ in the process, we obtain the claimed resolution. $\square$

Given the Cartan-Eilenberg resolution, we can evaluate using the global sections functor Γ_X to obtain a cochain complex of resolutions of $\mathcal{R}(X)$ -modules. When this complex is acyclic from the first row and column onwards, we have the following lemma:

Theorem 4.3.2 (De Rham-Weil). *Given a sheaf $\mathcal{M}$ on X and a resolution $\mathcal{A}^\cdot$ such that each A^n has trivial cohomology modules, there exists an isomorphism*

$$H^\cdot(X; \mathcal{M}) \cong H^\cdot(\mathcal{A}^\cdot)$$

Proof. The outline of the diagram chase in this proof was sketched by the supervisor and the details are filled in by the author. Given the resolution $\mathcal{A}^\cdot$, we can obtain a Cartan-Eilenberg resolution $\mathcal{J}^{\cdot,\cdot}$ by the previous lemma. Evaluating this resolution by the global sections functor Γ_X we obtain a complex of resolutions of $\mathcal{R}(X)$ -modules. The rows, excluding the first, are exact by Proposition 3.5.2(ii) and the columns, also excluding the first, are exact by our assumption that A^n has trivial sheaf cohomology and the fact the $\mathcal{J}^{\cdot,\cdot}$ are injective. We can now perform an elicit diagram chase to inductively construct a morphism $\varphi^\cdot : H^\cdot(X; \mathcal{M}) \rightarrow H^\cdot(\mathcal{A}^\cdot)$. We label the maps as in the diagram,

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \vdots \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 I^3 & \xrightarrow{\iota_{I_3}} & J^{3,0} & \xrightarrow{d_{3,0}^h} & J^{3,1} & \xrightarrow{d_{3,1}^h} & J^{3,2} \xrightarrow{d_{3,2}^h} J^{3,3} \longrightarrow \dots \\
 d_I^2 \uparrow & & d_{2,0}^v \uparrow & & d_{2,1}^v \uparrow & & d_{2,2}^v \uparrow \\
 I^2 & \xrightarrow{\iota_{I_2}} & J^{2,0} & \xrightarrow{d_{2,0}^h} & J^{2,1} & \xrightarrow{d_{2,1}^h} & J^{2,2} \xrightarrow{d_{2,2}^h} J^{2,3} \longrightarrow \dots \\
 d_I^1 \uparrow & & d_{1,0}^v \uparrow & & d_{1,1}^v \uparrow & & d_{1,2}^v \uparrow \\
 I^1 & \xrightarrow{\iota_{I_1}} & J^{1,0} & \xrightarrow{d_{1,0}^h} & J^{1,1} & \xrightarrow{d_{1,1}^h} & J^{1,2} \xrightarrow{d_{1,2}^h} J^{1,3} \longrightarrow \dots \\
 d_I^0 \uparrow & & d_{0,0}^v \uparrow & & d_{0,1}^v \uparrow & & d_{0,2}^v \uparrow \\
 I^0 & \xrightarrow{\iota_{I_0}} & J^{0,0} & \xrightarrow{d_{0,0}^h} & J^{0,1} & \xrightarrow{d_{0,1}^h} & J^{0,2} \xrightarrow{d_{0,2}^h} J^{0,3} \longrightarrow \dots \\
 \iota_{\mathcal{M}} \uparrow & & j_{A_0} \uparrow & & j_{A_1} \uparrow & & j_{A_2} \uparrow \\
 \mathcal{M} & \xrightarrow{j_{\mathcal{M}}} & A^0 & \xrightarrow{d_A^0} & A^1 & \xrightarrow{d_A^1} & A^2 \xrightarrow{d_A^2} A^3 \longrightarrow \dots
 \end{array}$$

First we prove the base case. Suppose $n = 0$, then $H^0(\mathcal{I}) = \ker d_I^0$. Let $x_0 \in H^0(\mathcal{I})$, then $x_0 \in \text{im } \iota_{\mathcal{M}}$. Since $\iota_{\mathcal{M}}(X)$ is injective, there exists a unique $a \in \mathcal{M}(X)$ such that $\iota_{\mathcal{M}}(X)(a) = x_0$. Now $j_{\mathcal{M}}(X)(a) \in \text{im } j_{\mathcal{M}}(X) = \ker d_A^0(X)$ so $j_{\mathcal{M}}(X)(a) \in H^0(\mathcal{A}^\cdot)$. So clearly, $H^0(\mathcal{I}) \cong H^0(\mathcal{A}^\cdot)$.

Now suppose $n > 0$,

There exists a $\varphi^n : H^n(X; \mathcal{M}) \rightarrow H^n(\mathcal{A}^\cdot)$

Let $[x_n] \in H^n(X; \mathcal{M}) = H^n(\mathcal{I})$, then there exists a $x'_n \in \ker d_I^n$ such that $x'_n \sim x_n$. By commutativity, $x'_n \in \ker d_I^n$ implies that $\iota_{I^n}(x'_n) \in \ker d_{n,0}^v = \text{im } d_{n-1,0}^v$. So there exists a $y_{n-1} \in J^{n-1,0}$ such that

$d_{n-1,0}^v(y_{n-1}) = \iota_{I^n}(x'_n)$. Again by commutativity,

$$x_{n-1} := d_{n-1,0}^h(y_{n-1}) \in \ker d_{n-1,1}^v.$$

Applying this argument $n-1$ times, we obtain $x_0 \in J^{0,n}$. Now since $x_0 \in \ker d_{0,n}^v = \text{im } J_{A^n}$ and J_{A^n} injective, there exists a unique $a \in A^n$ which is also in $\ker d_A^n$ by the commutativity (where in particular the injectivity of $J_{R^{n+1}}$ allows us to conclude this). We define $a := \varphi(x_n)$. As we have done before, e.g. in the snake lemma, we can argue that

$$\varphi(x_n + rx'_n) = \varphi(x_n) + r\varphi(x'_n)$$

for some $x_n, x'_n \in H^n(X; \mathcal{M})$ and $r \in \mathcal{R}(X)$, thus φ^n is an $\mathcal{R}(X)$ -modules homomorphism. To argue this φ^n is well-defined it suffices to show that $x_n \in \text{im } d_I^{n-1}$ implies that $\varphi(x_n) \in \text{im } d_A^{n-1}$. We can employ a similar argument as above. Let $x_n \in \text{im } d_I^{n-1}$ then there exists a $x'_{n-1} \in I^{n-1}$ whose image under d_I^{n-1} is x_n . Define

$$a_{n-1} := y_{n-1} - \iota_{I^{n-1}}(x_{n-1}) \in \ker d_{n-1,0}^v$$

(this follows by the commutativity of the diagram). By the exactness of this column, there exists a $k_{n-2} \in J^{n-2,0}$ that is a_{n-1} under $d_{n-2,0}^v$. Additionally, we can argue that $d_{n-1,0}^h(a_{n-1}) \in \ker d_{n-1,1}^v$. By commutativity,

$$d_{n-2,1}^v \circ d_{n-2,0}^h(k_{n-2}) = d_{n-1,0}^h(a_{n-1}),$$

therefore, for $y_{n-2} \in J^{n-2,1}$ as defined above, $d_{n-2,1}^v(y_{n-2}) = d_{n-1,0}^h(a_{n-1})$. Consequently, we can define and conclude that

$$a_{n-2} := y_{n-2} - d_{n-2,0}^h(k_{n-2}) \in \ker d_{n-2,1}^v$$

where

$$d_{n-2,1}^h(a_{n-2}) = d_{n-2,1}^h(y_{n-2}) = x_{n-2}.$$

Applying this step $n-1$ times, we obtain $a_0 \in J^{0,n-1}$ such that $a_0 \in \ker d_{0,n-1}^v$ and $d_{0,n-1}^h(a_0) = x_0$. Thus $a_0 \in \text{im } J_{A^{n-1}}$ and $x_0 \in \text{im } J_{A^n}$. So there exists a unique $b \in A^{n-1}$ such that $J_{A^{n-1}}(b) = a_0$. By commutativity, and injectivity of J_{A^n} , $d^{n-1}(b) = \varphi(x_n)$ and thus φ is well-defined.

$[\varphi^n]$ is an isomorphism

By the above argument, there exists a $\psi^n : H^n(A) \rightarrow H^n(X; \mathcal{M})$. Similar to the x_i, y_i obtained in evaluating $\varphi^n(x_n)$, for $\varphi(x_n) \in A^n$, we have $x'_k \in J^{n-k,k}$ such that $x'_{k-1} = d_{n-k-1,k}^v(y'_k) \in \ker d_{k,n-k}^h$ and $d_{n-k-1,k}^h(y'_k) = x'_k$ for $k < n$ and $d_{0,n-1}^h(y'_{n-1}) = J_{A^n}(\varphi(x_n))$. We want that $x_n - \psi^n \circ \varphi^n(x_n) \in \text{im } d_I^{n-1}$.

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & \uparrow & & \\
 \dots & \longrightarrow & J^{1,n-2} & \longrightarrow & J^{1,n-1} \\
 & & \uparrow & & \uparrow \\
 & & a_n \in J^{0,n-2} & \longrightarrow & J^{0,n-1} \longrightarrow J^{0,n} \\
 & & & & \uparrow \\
 & & & & A^{n-1} \longrightarrow A^n
 \end{array}$$

Note that $x'_n = x_0$. So $y'_n - y_0 \in \ker d_{0,n-1}^h$. So there exists an $a_{n-1} \in J^{0,n-2}$, such that by commutativity,

$(y'_{n-1} - y_1) - d_{0,n-2}^v(a_{n-1}) \in \ker d_{1,n-2}^h$. Hence we find an $a_{n-2} \in J^{1,n-3}$ such that $(y'_{n-2} - y_2) - d_{1,n-3}^v(a_{n-2}) \in \ker d_{2,n-3}^h$. Repeating this step, we eventually obtain $(y'_1 - y_{n-1}) - d_{n-2,0}^v(a_1) \in \ker d_{n-1,0}^h$.

$$\begin{array}{ccccc}
 I^n & \hookrightarrow & J^{n,0} & & \\
 \uparrow & & \uparrow & & \\
 I^{n-1} & \hookrightarrow & J^{n-1,0} & \longrightarrow & J^{n+1,1} \\
 & & \uparrow & & \uparrow \\
 & & J^{n-2,0} & \longrightarrow & J^{n-2,1} \longrightarrow \dots \\
 & & & & \downarrow \\
 & & & & \vdots
 \end{array}$$

So there exists an $a_0 \in I^{n-1}$ such that $\iota_{I^n} \circ d_I^{n-1}(a_0) = x'_0 - \iota_{I^n}(x_n)$. Hence, since ι_{I^n} is injective (noting $x'_0 = \iota_{I^n}(\psi^n \circ \varphi^n(x_0))$), $\psi^n \circ \varphi^n(x_n) - x_n \in \text{im } d_I^{n-1}$. By a similar argument, $[\varphi^n \circ \psi^n(x_n)] = [x_n]$, so $[\varphi^n]$ is an isomorphism, completing the proof. $\square$

Weibel mentions when introducing the snake lemma that its proof is best understood by attempting it on your own [8]. As demonstrated in the proof above, a compact diagram chase may go at the expense of readability. For a more convincing argument of the existence of the isomorphism, see Proposition 3.6.2 in [4], where Springer uses the long exact sequence in cohomology and the same trick we used to construct the Cartan-Eilenberg resolution (that $0 \rightarrow \ker d_R^n \rightarrow R^n \rightarrow \text{im } d_R^n \rightarrow 0$ is exact). Unfortunately, via this approach there is no explicit construction of the isomorphism. Alternatively, we could have also opted to use total cohomology of double complexes as defined in [8], which would have required more work.

4.4 Acyclic sheaves

Finding an injective resolution associated to a sheaf $\mathcal{M}$ may not be an easy task. In this section, we discuss various resolutions inducing isomorphic cohomologies when the space X satisfies certain properties. Each of these resolutions will relate by one important properties, the sheaves are all acyclic:

Definition 4.4.1 (Acyclic sheaf). A sheaf $\mathcal{M}$ on X is **acyclic** if $H^n(X; \mathcal{M}) = 0$ for all $n > 0$. We say a resolution of some sheaf $\mathcal{A}$ is acyclic if A^i is acyclic for all $i \in \mathbb{Z}_{\geq 0}$. $\diamond$

Therefore by the De Rham-Weil theorem, acyclic resolution and injective resolutions induce isomorphic cohomologies. Now we will study several classes of sheaves that turn out to be acyclic. First we define soft and fine sheaves. It turns out that fine sheaves have a connection to vector bundles.

4.4.1 Intermezzo: Paracompactness and Partitions of Unity

Before introducing fine sheaves, which are defined on paracompact spaces, we briefly recall paracompactness. This is a notion from topology about the open covers of a topology.

Definition 4.4.2 (Paracompact Space). A topological space X is **paracompact** if every open cover $\mathcal{U}$ admits a refinement $\tilde{\mathcal{U}}$ that is locally finite, i.e., for every $x \in X$, x intersects only a finite number of opens in $\tilde{\mathcal{U}}$. $\diamond$

Clearly then any compact space is paracompact. In fact any locally compact, second-countable Hausdorff space is paracompact:

Theorem 4.4.3. *Any locally compact, second-countable Hausdorff space is paracompact. In particular, any manifold is.*

Proof. The theorem follows by some elementary point-set topology.

Claim 1. If X is Hausdorff and locally compact then it admits a basis of precompact opens.

Proof by author. Let $x \in X$ and let U be any neighborhood of x . Since X is locally compact, x has a neighborhood V contained in some compact set. Now $x \in U \cap V$ and since $U \cap V$ is contained in some compact set $\overline{U \cap V}$ is also compact, so $U \cap V$ is precompact.

Claim 2. A locally compact, second-countable Hausdorff space X has an exhaustion by compact sets.

The proof of this claim is Proposition A.60 [2]. By the previous claim, X has a basis of precompact open subsets and by second-countability, X is covered by countably many of such sets. We prove by induction that there exists an exhaustion $\{K_i\}_{i=1}^\infty$. Let $\{U_i\}_{i=1}^\infty$ be the open cover and $K_1 = \overline{U_1}$. Assume we have compact sets $K_1 \subset \cdots \subset K_k$ such that $U_i \subset K_i$ and $K_j \subset \mathring{K}_{j+1}$ for all $0 \leq i \leq k$ and $0 \leq j < k$. Since K_k is compact, there exists some n_k such that $K_k \subset U_1 \cup \cdots \cup U_{n_k}$. Define $K_{k+1} = \overline{U_1} \cup \cdots \cup \overline{U_{n_k}}$, since we want $U_{k+1} \subset K_{k+1}$, choose $n_k \geq k + 1$. As a finite union of compacts, K_{k+1} is compact and clearly $K_k \subset \mathring{K}_{k+1}$.

The proof of the theorem is as in theorem 1.15 in [2]. Let $\mathcal{U}$ be any open cover and $\mathcal{B}$ be any basis of topology of X . By assumption, claim 2 implies that there exists an exhaustion by compacts $\{K_j\}_{j=1}^\infty$ of X . Using this exhaustion we define,

$$V_j := K_{j+1} \setminus \mathring{K}_j, \quad W_j = \mathring{K}_{j+2} \setminus K_{j-1}.$$

Note that V_j is compact as a closed subset of K_{j+1} and W_j is open with $V_j \subset W_j$. Let $j \in \mathbb{Z}_{>0}$, for any $x \in V_j$ there exists an open neighborhood of x , $U_x \in \mathcal{U}$. Also there exists a $B_x \in \mathcal{B}$ such that $x \in B_x \subset U_x \cap W_j$. These B_x form an open cover of V_j which by compactness refines to a finite subcover $\mathcal{V}_j$. The concatenation of these finite subcovers forms a countable cover of X which is a refinement of $\mathcal{U}$, say $\tilde{\mathcal{U}}$. Now to prove this $\tilde{\mathcal{U}}$ is locally finite. Since for each cover $\mathcal{V}_j$ of V_j , the sets are in W_j , and $W_j \cap W_{j'} = \emptyset$ for all j, j' such that $j' < j - 2$ or $j' > j + 2$, each $x \in V_j$ is contained in finitely many opens in $\tilde{\mathcal{U}}$, since the V_j cover X , the result follows. The final claim of the theorem follows by the fact that manifolds are locally compact (Proposition 1.12 in [2]). $\square$

Also recall the following theorem, which we use without proof.

Theorem 4.4.4 (Theorem 1.17 [1]). *If X is a paracompact Hausdorff space and $A \subset C(X)$ is closed under locally finite sums, quotients and is normal, there exists for every open cover $\mathcal{U}$ of X an A -valued partition of unity subordinated to $\mathcal{U}$.*

In the following section we introduce fine sheaves, the definitions and proofs are taken from [11].

4.4.2 Fine Sheaves

We assume X to be a paracompact Hausdorff space (e.g. any manifold satisfies these conditions by theorem 4.4.3). A fine sheaf $\mathcal{F}$ of $\mathcal{R}$ -modules on X is a sheaf that admits an $\mathcal{R}(X)$ -valued partition of unity

subordinate to any locally finite open cover $\mathcal{U}$ of X . To be able to formally define this, we need to define supports,

Definition 4.4.5. Considering the sheaf $\mathcal{F}$, the **support** of a section $s \in \mathcal{F}(U)$ for any open $U \subset X$ is the closed set $\{x \in U : s_x \neq 0\}$. $\diamond$

To see that the support is indeed a closed set, let X be a topological space and $\mathcal{F}$ a sheaf on X . Let $U \subset X$ and $s \in \mathcal{F}(U)$. For some $x \in X \setminus \text{supp } s$, $s_x = 0$. Yet since $s_x \in \mathcal{F}_x$ the equivalence relation implies that there exists some open neighborhood of x , $V \subset U$ such that $\rho_V^U(s) = 0$, so since $s_y = 0$ for all $y \in V$ implies that $V \subset X \setminus \text{supp } s$, the support is closed.

Definition 4.4.6 (Proposition 12 [11]). A sheaf of $\mathcal{R}$ -modules $\mathcal{M}$ on a paracompact Hausdorff space X is **fine** if for every locally finite open cover $\mathcal{U}$ there exist $\lambda_i \in \mathcal{R}(X)$ such that:

- i. $\text{supp } \lambda_i \subset U_i$ for every $i \in I$
- ii. $\sum_{i \in I} \lambda_i = 1$.

$\diamond$

For an alternative approach to fine sheaves see [5].

Example 4.4.7 (Sheaf of sections of a vector bundle). We show that the sheaf of sections of an $\mathbb{R}$ -valued vector bundle (E, X, π) is fine. By lemma 2.3.2, this boils down to the existence of $C^\infty(X)$ -partitions of unity subordinate to a locally finite cover. By theorem 4.4.4, since $C^\infty(X) \subset C(X)$ is normal, closed under finite sums and quotients, the sheaf of sections of a vector bundle admits a $C^\infty(X)$ -partition of unity subordinate to any locally finite open cover $\mathcal{U}$. Thus this class of sheaves is fine. $\triangle$

Proposition 4.4.8. *Fine sheaves are acyclic.*

Proof. The proof is Proposition 13 from [11], we chose to go with this proof due to its compactness, for another proof via soft sheaves see [5]. Consider a fine sheaf $\mathcal{F}$ of $\mathcal{R}$ -modules, and resolve $\mathcal{F}$ by an injective resolution $\mathcal{I}$. We claim that $\ker d_I^n(X)/\text{im } (d_I^{n-1}(X)) = 0$ for all $n > 0$. Let $n \in \mathbb{Z}_{>0}$ and $s \in I^n(X)$ such that $d_I^n s = 0$. By exactness, $s \in (\text{im } d_I^{n-1})(X)$ so by Lemma 3.2.5, for each $x \in X$ there exists an open neighborhood of x , U_x such that $\rho_{U_x}^X(s) \in \text{im } (d_I^{n-1}(U_x))$ so there exists a $t_x \in I^{n-1}(U_x)$ such that $d_I^{n-1}(t_x) = \rho_{U_x}^X(s)$. Let $\mathcal{U} := \{U_x\}_{x \in X}$ be the open cover of X by such U_x . Since X is paracompact, there is a locally finite refinement of $\mathcal{U}$, $\tilde{\mathcal{U}} = \{U_i\}_{i \in I}$, with corresponding $t_i \in I^{n-1}(U_i)$ such that $d_I^{n-1}(t_i) = \rho_{U_i}^X(s)$.

Since $\mathcal{F}$ is fine, there exists an $\mathcal{R}(X)$ -valued partition of unity subordinate to $\tilde{\mathcal{U}}$, $\lambda_i \in \mathcal{R}(X)$. Given $a \in \mathcal{R}(X)$ with $\text{supp } a \subset U$ and $e \in I^{n-1}(U)$, we define $\overline{ae}$ as the amalgamation of $\rho_U^X(a)e$ and $0 \in I^{n-1}(X \setminus \text{supp } a)$. Using this notation and the fact that $\tilde{\mathcal{U}}$ is locally finite,

$$\begin{aligned} d_I^{n-1}(X)(\sum_{i \in I} \overline{\lambda_i t_i}) &= \sum_{i \in I} d_I^{n-1}(X)(\overline{\lambda_i t_i}) \\ &= \sum_{i \in I} \overline{\rho_{U_i}^X(\lambda_i) d_I^{n-1}(U_i)(t_i)} = \sum_{i \in I} \overline{\rho_{U_i}^X(\lambda_i) \rho_{U_i}^X(s)} \\ &= \sum_{i \in I} \overline{\rho_{U_i}^X(\lambda_i s)} = \sum_{i \in I} \lambda_i s = \left(\sum_{i \in I} \lambda_i \right) s = s \end{aligned}$$

Where in equality $*$, we use the fact that $\overline{\rho_{U_i}^X(\lambda_i s)} = \lambda_i s$ since $\text{supp } \lambda_i \subset U_i$. Hence we conclude that $s \in \text{im } d_I^{n-1}$ when $s \in \ker d_I^n$, proving the result. $\square$

4.4.3 Flabby sheaves

Another class of sheaves we will use in this thesis are flabby sheaves. Later on we will see that the singular cohomology sheaf is flabby. The results in this section and their proofs are from [5]. To begin with we define what it means for a sheaf to be flabby:

Definition 4.4.9. A sheaf $\mathcal{F}$ on X is flabby if for every open $U \subset X$, the restriction map $\rho_U^X : \mathcal{F}(X) \rightarrow \mathcal{F}(U)$ is surjective. $\diamond$

Example 4.4.10. • As we have seen as a claim in the proof of the long exact sequence of cohomology, any injective sheaf is flabby.

- The Godement sheaf $G_{\mathcal{P}}$ is flabby. Recall that, for any $U \subset X$ open, the restriction map was

$$\rho_U^X : \prod_{x \in X} \mathcal{P}_x \rightarrow \prod_{x \in U} \mathcal{P}_x, (s_x)_{x \in X} \rightarrow (s_x)_{x \in U}$$

which is clearly surjective. $\triangle$

We will prove that flabby sheaves are acyclic, we will use the following result:

Lemma 4.4.11 (Lemma 5.1.4 [5]). *If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is exact and A is flabby, then it is exact after evaluation by the global sections functor Γ_X .*

Proof. This statement is of a similar nature to claim 2 in the proof of Theorem 4.2.1. In fact we can use the same reasoning as in claim 2, since the only difference is that in the Zorn's lemma argument we already know directly that A is flabby. $\square$

Now we prove the final result of this chapter, the proof is an application of Theorem 4.2.1.

Proposition 4.4.12 (Proposition 5.1.3 [5]). *A flabby sheaf is acyclic.*

Proof. We follow the proof of Proposition 5.1.3 in [5]. Let $\mathcal{F}$ be a flabby sheaf over topological space X , and let $\mathcal{I}^\bullet$ be an injective resolution. Then for $\mathcal{M} = \text{im}(\mathcal{I}^0 \rightarrow \mathcal{I}^1)$, we obtain a short exact sequence,

$$0 \rightarrow \mathcal{F} \xrightarrow{\psi} \mathcal{I}^0 \xrightarrow{\varphi} \mathcal{M} \rightarrow 0.$$

First we claim that $\mathcal{M}$ is flabby (for this part we follow the proof of Lemma 5.1.5 [5]). $\mathcal{F}$ is flabby and $\mathcal{I}^0$ is flabby by claim 1 in the proof of Theorem 4.2.1. It is easy to see that then the restriction sheaves $\mathcal{F}|_U$ and $\mathcal{I}^0|_U$ are also flabby for any open $U \subset X$. Also by lemma 4.4.11 and the fact that $\Gamma_X \mathcal{F}|_U = \mathcal{F}(U)$, the sequence

$$0 \rightarrow \mathcal{F}(U) \xrightarrow{\psi(U)} \mathcal{I}^0(U) \xrightarrow{\varphi(U)} \mathcal{M}(U) \rightarrow 0$$

is exact. Now let $m \in \mathcal{M}(U)$ then by exactness there exists an $n \in \mathcal{I}^0(U)$ such that $\varphi(U)(n) = m$. Since $\mathcal{I}^0$ is flabby, there exists a $\tilde{n} \in \mathcal{I}^0(X)$ such that $\rho_U^X(\tilde{n}) = n$. Hence,

$$m = \varphi(U)(n) = \varphi(U)(\rho_U^X(\tilde{n})) = \rho_U^X(\varphi(X)(\tilde{n}))$$

thus $\rho_U^X : \mathcal{M}(X) \rightarrow \mathcal{M}(U)$ is surjective, proving that $\mathcal{M}$ is flabby.

Now by theorem 4.2.1 we obtain a long exact sequence in cohomology,

$$\begin{array}{c}
0 \longrightarrow H^0(X; \mathcal{F}) \rightarrow H^0(X; I^0) \rightarrow H^0(X; \mathcal{M}) \xrightarrow{\partial^0} H^1(X; \mathcal{F}) \longrightarrow \dots \\
\downarrow \hspace{15em} \downarrow \\
\longrightarrow \dots \longrightarrow H^{n-1}(X; \mathcal{M}) \rightarrow H^n(X; \mathcal{F}) \rightarrow H^n(X; I^0) \longrightarrow \dots
\end{array}$$

We will now use the exactness of the sequence to obtain our desired result. First we prove that $H^1(X; \mathcal{F}) = 0$. Since $\mathcal{F}$ is flabby, we find by Lemma 4.4.11 that,

$$0 \rightarrow \Gamma_X \mathcal{F} \rightarrow \Gamma_X I^0 \rightarrow \Gamma_X \mathcal{M} \rightarrow 0 \quad (4.1)$$

is exact. Hence, $\text{im}(\partial^0 : H^0(X; \mathcal{M}) \rightarrow H^1(X; \mathcal{F})) = 0$ since by proposition 3.5.2(ii), the global sections are isomorphic to the 0-th cohomology groups. By exactness of the long exact sequence in cohomology, the kernel of the morphism $H^1(X; \mathcal{F}) \rightarrow H^1(X; I^0)$ is trivial. Hence by the first isomorphism theorem for modules, $H^1(X; \mathcal{F}) \cong H^1(X; I^0)$. By proposition 3.5.2(i), I^0 is acyclic (since it is injective). Consequently, $H^1(X; \mathcal{F}) = 0$.

Now assume by strong induction that for some $n \in \mathbb{N}$, all flabby sheaves $\mathcal{F}$ and all $k < n$, $H^k(X; \mathcal{F}) = 0$. Then in particular, since $\mathcal{M}$ is flabby, $H^{k-1}(X; \mathcal{M}) = 0$. Since I^0 is acyclic, we also have $H^k(X; I^0) = 0$. So by our long exact sequence in cohomology,

$$0 \rightarrow H^k(X; \mathcal{F}) \rightarrow 0$$

is exact. Therefore $H^k(X; \mathcal{F}) = 0$. □

5 Cohomology Theories

In this section, we apply the tools introduced in this thesis to establishing an equivalence of cohomology theories via sheaf cohomology. In particular, the aim of this chapter is to provide a description of the proof of de Rham's theorem via sheaf cohomology.

5.1 De Rham Cohomology

Let X be a manifold. In Example 2.3.4, we saw that the k -forms corresponded to maps

$$\omega : X \ni p \mapsto \omega_p \in \bigwedge^k T_p^* X.$$

We denote the set of all k -forms over X by $\Omega^k(X)$. These sets along with the differentials (for details see [1]),

$$d^k : \Omega^k(X) \rightarrow \Omega^{k+1}(X).$$

We define the **De Rham complex** as,

$$\Omega^0(X) \xrightarrow{d^0} \Omega^1(X) \xrightarrow{d^1} \Omega^2(X) \rightarrow \dots$$

The **De Rham cohomology modules** are the cohomology modules of this complex. Note also that De Rham cohomology is an invariant for manifolds.

Definition 5.1.1. Let X, Y be manifolds and $f, g : X \rightarrow Y$ be smooth maps. These are said to be **smoothly homotopic** if there exists a smooth homotopy

$$F : X \times [0, 1] \rightarrow Y$$

such that $F(-, 0) = f(-)$ and $F(-, 1) = g(-)$. A **smooth homotopy equivalence** is a homotopy equivalence existing out of smooth homotopies. $\diamond$

Theorem 5.1.2 (Homotopy Invariance). *Let X, Y be manifolds. If $f, g : X \rightarrow Y$ are smoothly homotopic maps, then they induce the same homomorphisms in cohomology. Furthermore, if f is a smooth homotopy equivalence, then X and Y have isomorphic De Rham cohomologies.*

Proof. See Theorem 5.41 [1]. $\square$

We can relate De Rham cohomology to the cohomology of a sheaf.

5.1.1 Sheaf Cohomological Version

Let X be a manifold and $\mathbb{R}$ be the constant sheaf on X , Ω^k the sheaves of k -forms on X . Recall that in Example 3.2.7 we have defined the differentials so that,

$$0 \rightarrow \mathbb{R} \hookrightarrow \Omega^0$$

defines cochain complex, which when evaluated under the global sections functor corresponds to the standard De Rham cochain complex whose cohomology groups we denote by $H_{\text{dR}}^k(X, \mathbb{R})$.

Lemma 5.1.3. *Let X , $\mathbb{R}$ and $\Omega^\cdot$ be as above. The de Rham complex*

$$0 \rightarrow \mathbb{R} \hookrightarrow \Omega^\cdot$$

is exact. The sheaves Ω^k are fine, hence this is an acyclic resolution of $\mathbb{R}$

Proof. By the Poincaré lemma, since X is a manifold, and hence locally euclidean, for every $x \in X$ there exists a neighborhood U_x such that $0 \rightarrow \mathbb{R}(U_x) \hookrightarrow \Omega^\cdot(U_x)$ is exact. From this, we deduce that the complex of sheaves is exact in the stalks. Hence by Theorem 3.2.4, we conclude that the complex is exact. In Example 4.4.7, we outlined how the existence of partitions of $C^\infty(X)$ partitions of unity implies that the sheaves Ω^k are fine, hence by Theorem 4.4.8, this resolution is acyclic. $\square$

Corollary 5.1.4. *There exists an isomorphism of cochain complexes,*

$$H_{dR}(X, \mathbb{R}) \cong H^\cdot(X; \mathbb{R})$$

Proof. By the previous lemma, $\Omega^\cdot$ is an acyclic resolution of $\mathbb{R}$, and acyclic resolutions induce isomorphic cohomologies to the sheaf cohomology by the De Rham-Weil theorem (Theorem 4.3.2). $\square$

5.2 Singular Cohomology

In this section we will define the singular sheaf cohomology. We follow [12] which gives a more comprehensive definition than in [4] and relates to the definition in [2]. We define singular cohomology by means of the dual concept.

5.2.1 Singular Homology

We give a brief introduction of singular homology as covered in [2] with some details from [13]. First we give some definitions.

Definition 5.2.1 (Geometric p-simplex). We list some definitions related to geometric simplices.

(Geometric simplex) A **geometric p-simplex** is the subset,

$$[v_0, \dots, v_p] = \left\{ \sum_{i=0}^p t_i v_i : 0 \leq t_i \leq 1, \sum_{i=0}^p t_i = 1 \right\} \subset \mathbb{R}^n.$$

Where p is the **dimension** and $v_0, \dots, v_p \in \mathbb{R}^n$ are the vertices. The coefficients $t_0, \dots, t_p$ are called the **barycentric coordinates** of point $\sum_{i=0}^p t_i v_i \in [v_0, \dots, v_p]$.

(Faces) **Faces** are nonempty simplices whose vertices are contained in $\{v_0, \dots, v_p\}$ and **boundary faces** are the $p-1$ -dimensional faces. There are $p+1$ of these and the i -th boundary faces is denoted $\partial_i[v_0, \dots, v_p] = [v_0, \dots, \hat{v}_i, \dots, v_p]$ (where the v_i is omitted).

(Standard Simplex) The **standard p-simplex** is denoted by $\Delta_p := [0, e_1, \dots, e_p] \subset \mathbb{R}^n$ where $e_1, \dots, e_p$ are the standard basis vectors in $\mathbb{R}^n$. Let X be a topological space. A continuous map $\sigma : \Delta_p \rightarrow X$ is called a **singular p-simplex in X** .

(Singular chain) For any topological space X , $C_p(X)$ is the free R module generated by the singular simplices and called the **singular chain module of X in degree p** . An element $\sigma \in C_p(X)$ is called a **singular p chain**.

(Affine Singular Simplex) In the case that $X \subset \mathbb{R}^n$ convex, for any $w_0, \dots, w_p \in X$, there exists a unique affine map $\varphi : \mathbb{R}^p \rightarrow \mathbb{R}^n$ generated by $e_i \mapsto w_i$. Then $A(w_0, \dots, w_p) := \varphi|_{\Delta_p}$ is called the **affine singular simplex in X** .

(Face Maps) The **face maps** in Δ_p are the $p+1$ affine $p-1$ -singular simplices in X , denoted $F_{i,p} : \Delta_{p-1} \rightarrow \Delta_p$ and induced by $e_0 \mapsto e_0, \dots, e_{i-1} \mapsto e_{i-1}, e_i \mapsto e_{i+1}, \dots, e_{p-1} \mapsto e_p$, and the domain is isomorphic to $\partial_i \Delta_p$.

◇

Definition 5.2.2 (Singular Boundary Operator). The **singular boundary operator** $\partial_n : C_p(X) \rightarrow C_{p-1}(X)$ is the R -module homomorphism induced by the maps on basis elements $\sigma \in C_p(X)$,

$$\partial_n(\sigma) = \sum_{i=0}^n (-1)^i \sigma \circ F_{i,p}.$$

◇

We can in fact prove that the concatenation of these boundary operators forms a chain complex

$$\cdots \rightarrow C_n(X) \rightarrow C_{n-1}(X) \rightarrow \cdots \rightarrow C_0(X).$$

Lemma 5.2.3. *The composition of boundary operators is trivial, i.e. $\partial^{n-1} \circ \partial^n = 0$.*

Proof. We follow the outline sketched in Lemma 18.2 [2]. This is just a simple computation, first off assuming $j > i$,

$$\begin{aligned} F_{j,n+1} \circ F_{i,n}(\Delta_{n-1}) &= F_{j,n+1}(\partial_i \Delta_n) \\ &= \partial_i \partial_j \Delta_{n+1} = \partial_{j-1} \partial_i \Delta_{n+1} \\ &= F_{i,n+1} \circ F_{j-1,n}(\Delta_{n-1}) \end{aligned}$$

where the second to last equality follows by $j > i$ implying that the i -th face of the j -th face of Δ_{n+1} is the $j-1$ -th face of the i -th face of Δ_{n+1} . Hence giving us the formula,

$$F_{j,n+1} \circ F_{i,n} = F_{i,n+1} \circ F_{j-1,n}. \quad (5.1)$$

Now let $c \in C_{n+1}(X)$. Writing out the summation and using some simple manipulations we obtain,

$$\begin{aligned} \partial^{n-1} \circ \partial^n(c) &= \sum_{i=0}^{n-1} (-1)^i \partial^n(c) \circ F_{i,n} = \sum_{i=0}^{n-1} (-1)^i \left(\sum_{j=0}^n (-1)^j c \circ F_{j,n+1} \right) \circ F_{i,n} \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^n (-1)^{i+j} c \circ F_{j,n+1} \circ F_{i,n} \\ &= \sum_{i=0}^{n-1} \left(\sum_{j=i+1}^n (-1)^{i+j} c \circ F_{j,n+1} \circ F_{i,n} + \sum_{j=0}^i (-1)^{i+j} c \circ F_{j,n+1} \circ F_{i,n} \right). \end{aligned}$$

Using formula (1) we can rewrite the left summand as follows,

$$\begin{aligned} \sum_{i=0}^{n-1} \sum_{j=i+1}^n (-1)^{i+j} c \circ F_{j,n+1} \circ F_{i,n} &\stackrel{(1)}{=} \sum_{i=0}^{n-1} \sum_{j=i+1}^n (-1)^{i+j} c \circ F_{i,n+1} \circ F_{j-1,n} \\ &= \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (-1)^{i+j+1} c \circ F_{i,n+1} \circ F_{j,n} \\ &= \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} (-1)^{j+i+1} c \circ F_{j,n+1} \circ F_{i,n} \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^i (-1)^{j+i+1} c \circ F_{j,n+1} \circ F_{i,n}. \end{aligned}$$

Now factoring out -1 this is precisely the negative of the right summand. Thus these two cancel out and hence $\partial_{n-1} \circ \partial_n(c) = 0$, completing the proof. $\square$

In light of this, homology being the dual to cohomology we can define boundaries and cycles.

Definition 5.2.4. We call a singular p -chain $c : \Delta_p \rightarrow X$ a **p -cycle**, denoting the set of p -cycles by $Z_p(X)$. If $\partial^p(c) = 0$ and a **p -boundary** if there exists a $p+1$ -chain b such that $c = \partial^{p+1}(b)$, the set of such chains is denoted $B_p(X)$. Using this we can define the p -th homology as,

$$H_p(X) = Z_p(X)/B_p(X)$$

the quotient module well-defined since by the previous lemma $B_p(X) \subset Z_p(X)$ is a submodule. $\diamond$

5.2.2 Homotopy Invariance of Singular Homology

We paraphrase the section on homotopy invariance of singular homology in Hatcher [13]. Although this is a bit of a tangent from the main topic of this thesis, we chose to include this as another application of some homological in the dual category of chain complexes. The goal of this section is to show that the singular homology of a contractible space is trivial, this result will play an important role in proving the exactness of a resolution by singular cohomology sheaves.

Lemma 5.2.5. *A continuous function $f : X \rightarrow Y$ induces a chain homotopy $f_{\#} : C_*(X) \rightarrow C_*(Y)$*

Proof. Define $f_{\#}^n$ to be the R -module homomorphism induced by the mappings of singular n -simplices

$\sigma : \Delta_n \rightarrow X$, $\sigma \mapsto f \circ \sigma$. The $f_{\#}$ commute with ∂ . as for all $\sigma \in C_{n+1}(X)$,

$$\begin{aligned} f_{\#}^n \circ \partial_{n+1}(\sigma) &= f_{\#}^n \left(\sum_{i=0}^{n+1} (-1)^i \sigma \circ F_{i,n+1} \right) = \sum_{i=0}^{n+1} (-1)^i f \circ \sigma \circ F_{i,n+1} \\ &= \partial_n(f \circ \sigma) = \partial_n \circ f_{\#}^{n+1}(\sigma). \end{aligned}$$

Hence $f_{\#}$ is a chain map. □

As with cochain maps, we can argue that chain maps induce homomorphisms on the homology modules. Namely, let α be a cycle. Then since $f_{\#}$ is a chain map, $\partial(f_{\#}(\alpha)) = f_{\#}(\partial\alpha) = 0$. By the same reasoning we can conclude that if β is a boundary such that $\beta = \partial\alpha$ for some α , then $f_{\#}(\beta) = \partial f_{\#}(\alpha)$ is a boundary. We denote the induced homomorphism $f_* : H_n(X) \rightarrow H_n(Y)$.

Theorem 5.2.6 (Homotopy invariance). *Singular homology is invariant under homotopy*

Proof. The proof is of a similar nature to the one we saw for the homotopy invariance of cohomology. The essence is to prove that two homotopic functions $f, g : X \rightarrow Y$ induce chain homotopic maps $f_{\#}, g_{\#}$. Given a homotopy $F : X \times [0, 1] \rightarrow Y$ between f and g , we define the operator,

$$P_n : C_n(X) \rightarrow C_{n+1}(Y), \sigma \mapsto \sum_{i=0}^n (-1)^i F \circ (\sigma \times \text{id}_{[0,1]})|_{[v_0, \dots, v_i, w_i, \dots, w_n]}.$$

This operator is well-defined since the $n+1$ -simplices $[v_0, \dots, v_i, w_i, \dots, w_n]$ given maps $\sigma \times \text{id}_{[0,1]}|_{[v_0, \dots, v_i, w_i, \dots, w_n]} : \Delta_{n+1} \rightarrow Y \in C_{n+1}(Y)$. Example ?? gives some intuition of what these simplices are.

Claim 1. $f_{\#}$ and $g_{\#}$ are chain homotopic, that is, $\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n = g_{\#}^{n+1} - f_{\#}^{n+1}$.

Let $\sigma \in C_n(X)$ by some singular n -chain. Note that,

$$\partial_n \circ P_n(\sigma) = \sum_{i=0}^n (-1)^i P_n(\sigma) \circ F_{i,n+1} = \sum_{i=0}^n \sum_{j=0}^n (-1)^{i+j} F \circ (\sigma \times \text{id}_{[0,1]})|_{[v_0, \dots, v_i, w_i, \dots, w_n]} \circ F_{j,n+1}.$$

We can split this into two summations,

$$\begin{aligned} &\sum_{i=0}^n \sum_{j=0}^i (-1)^{i+j} F \circ (\sigma \times \text{id}_{[0,1]})|_{[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n]} \\ &+ \sum_{j=i}^n (-1)^{i+j+1} F \circ (\sigma \times \text{id}_{[0,1]})|_{[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]}. \end{aligned}$$

Now in the cases were $i = j$ the terms of the left summation

$$(-1)^{i+1+i+1} F \circ (\sigma \times \text{id}_{[0,1]})|_{[v_0, \dots, v_i, \hat{v}_{i+1}, w_{i+1}, \dots, w_n]}$$

cancel with the terms

$$(-1)^{i+i+1} F \circ (\sigma \times \text{id}_{[0,1]})|_{[v_0, \dots, v_i, \hat{w}_i, w_{i+1}, \dots, w_n]}.$$

Leaving the terms

$$F \circ (\sigma \times \text{id}_{[0,1]})|_{[\hat{v}_0, w_0, \dots, w_n]} = F \circ (\sigma|_{\Delta_n} \times \text{id}_{[0,1]}|_{\{0\}}) = g \circ \sigma = g_{\#}^{n+1}(\sigma)$$

and

$$-F \circ (\sigma \times \text{id}_{[0,1]})| [v_0, \dots, v_n, \hat{w}_n] = -F \circ (\sigma|_{\Delta_n} \circ \text{id}_{[0,1]}|_{\{1\}}) = -f \circ \sigma = -f_{\#}^{n+1}(\sigma).$$

Note that,

$$P_{n-1} \circ \partial_n(\sigma) = P_{n-1} \left(\sum_{j=0}^n (-1)^j \sigma \circ F_{j,n} \right) = \sum_{j=0}^n (-1)^j P_{n-1}(\sigma| [e_0, \dots, \hat{e}_j, \dots, e_n])$$

Since each $\sigma| [e_0, \dots, \hat{e}_j, \dots, e_n] : \partial_i \Delta_n \times [0, 1] \rightarrow X \times [0, 1]$ we have a choice of vertices, $v_0, \dots, \hat{v}_j, \dots, v_n$ and $w_0, \dots, \hat{w}_j, \dots, w_n$. Where again $\hat{v}_j$ denotes v_j being omitted. Hence,

$$\begin{aligned} P_{n-1}(\sigma| [e_0, \dots, \hat{e}_j, \dots, e_n]) &= \sum_{i=0}^{j-1} (-1)^i F \circ (\sigma \times \text{id}_{[0,1]})| [v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n] \\ &\quad + \sum_{i=j}^{n-1} (-1)^i F \circ (\sigma \times \text{id}_{[0,1]})| [v_0, \dots, \hat{v}_j, \dots, v_{i+1}, w_{i+1}, \dots, w_n]. \end{aligned}$$

Which can be rewritten as,

$$\begin{aligned} P_{n-1}(\sigma| [e_0, \dots, \hat{e}_j, \dots, e_n]) &= \sum_{i=0}^{j-1} (-1)^i F \circ (\sigma \times \text{id}_{[0,1]})| [v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n] \\ &\quad + \sum_{i=j+1}^n (-1)^{i-1} F \circ (\sigma \times \text{id}_{[0,1]})| [v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n]. \end{aligned}$$

Thus we obtain the expression $\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n = g_{\#}^{n+1} - f_{\#}^{n+1}$ and so $f_{\#}$ and $g_{\#}$ are chain homotopic.

Now to show $f_{\#}^n$ and $g_{\#}^n$ are the same. Let $\alpha \in C_n(X)$ by a cycle, since

$$g_{\#}^n(\alpha) - f_{\#}^n(\alpha) = \partial_{n+1} \circ P_n(\alpha) + P_{n-1} \circ \partial_n(\alpha) = \partial_{n+1} \circ P_n(\alpha),$$

cycles are sent to boundaries, thus $g_{\#}^n([\alpha]) = f_{\#}^n([\alpha])$.

Similarly to cochain maps, chain maps satisfy the operations, $(f \circ g)_{\#}^n = f_{\#}^n \circ g_{\#}^n$ and $(\text{id}_X)_{\#}^n = \text{id}_{H_n(X)}$ for all n . Therefore, if $f : X \rightarrow Y$ is a homotopy equivalence, since there exists a $g : Y \rightarrow X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X . We can conclude that, $f_{\#}^n \circ g_{\#}^n = \text{id}_{H_n(Y)}$ and $g_{\#}^n \circ f_{\#}^n = \text{id}_{H_n(X)}$ thus that $H_n(X) \cong H_n(Y)$ for all n . $\square$

A space X is contractible when it is homotopy equivalent to a point. The following is a consequence of the theorem for contractible spaces.

Corollary 5.2.7. *A contractible space X gives us a trivial singular homology.*

Proof. Well by the theorem, $H_n(X) \cong H_n(\{p\})$ for some point p . $C_n(\{p\})$ is the free module $R[\Delta_n \rightarrow \{p\}] = R[p_n]$. So for n uneven, $\partial_n(rp_n) = rp_{n-1}$, while for n even, $\partial_n(rp_n) = 0$. Thus $H_n(\{p\}) = 0$. $\square$

5.2.3 Singular Cohomology

In this section we define singular cohomology as covered in e.g. [2]. We consider the singular cohomology of the vector space $\mathbb{R}$ although this can be generalized to modules.

Definition 5.2.8. Given a topological space X , define $C^p(X) = \text{Hom}(C_p(X), \mathbb{R})$. Clearly $C^p(X)$ admits a vectorspace under point-wise addition and scalar multiplication,

$$\begin{aligned}(f + g)(x) &:= f(x) + g(x), \quad f, g \in S_{\Delta_p}(U), x \in C_p(U) \\ (r \cdot f)(x) &= r \cdot f(x), \quad f \in S_{\Delta_p}(U), x \in C_p(U).\end{aligned}$$

◇

Definition 5.2.9 (Singular Coboundary Operator). Using that $\text{Hom}(-, \mathbb{R})$ is a contravariant functor, we can define the **singular coboundary operator** as,

$$d^n = \text{Hom}(\partial^{n+1}, \mathbb{R}) : \text{Hom}(C_n(X), \mathbb{R}) \rightarrow \text{Hom}(C_{n+1}(X), \mathbb{R}), \sigma \mapsto (-1)^{n+1} \sigma \circ \partial_{n+1}.$$

This is clearly a module homomorphism.

◇

Remark 5.2.10. Since for $d^{n+1} \circ d^n = \text{Hom}(\partial_{n+2}, \mathbb{R}) \circ \text{Hom}(\partial_{n+1}, \mathbb{R})$, the function maps

$$\sigma \mapsto (-1)^{n+1} \cdot (-1)^n \sigma \circ \partial_{n+2} \circ \partial_{n+1} = 0$$

we have that $d^n \circ d^{n+1} = 0$. Hence,

$$0 \rightarrow C^0(X) \xrightarrow{d^0} C^1(X) \rightarrow \dots$$

is a cochain complex of presheaves. In particular the dual is exact if the original is.

◇

Definition 5.2.11 (Singular Cohomology). In light of the above discussion, the **n-th singular cohomology module** is defined as,

$$H_{\text{sing}}^n(X, \mathbb{R}) = H^n(C(X)).$$

◇

In addition, since we defined singular cohomology via singular homology, we can extend our results from singular homology to this new situation. In particular,

Theorem 5.2.12 (Homotopy Invariance). *Singular cohomology is homotopy invariant.*

Proof. Follows easily from the homotopy invariance of singular homology, as mentioned in [2].

□

5.2.4 Sheaf-Theoretic Singular Cohomology

As we develop this for the proof of the De Rham theorem, let X be a smooth manifold and $\mathcal{M} = \mathbb{R}$. In this section we follow [12]. We will start by finding a resolution of $\mathbb{R}$ that gives us a cohomology isomorphism to the singular cohomology and then show this is indeed the case.

Definition 5.2.13 (Singular Cohomology Presheaf). Define $C^p : U \mapsto \text{Hom}(C_p(U), \mathbb{R}) = (C_p(U))^*$ to be the singular cohomology presheaf. Note that for opens $U \subset V \subset X$ there is an obvious inclusion map $\iota_U^V : C_p(U) \hookrightarrow C_p(V)$ defined by the fact that any p -simplex on U is also a p -simplex on V under the inclusion $U \subset V$. These induce the restriction maps,

$$\rho_U^V : C^p(V) \rightarrow C^p(U), f \mapsto f \circ \iota$$

on C^p . ◇

Definition 5.2.14 (Singular Cohomology Sheaf). The sheafification of the presheaf C^n gives us the **singular cohomology sheaf**, $\mathcal{C}^n$. ◇

Lemma 5.2.15. *When X is a smooth manifold, $\mathcal{C}$ is a resolution of $\mathbb{R}$.*

Proof. For the first part of the proof we follow Remark 10.67 [12]. For any $U \subset X$, $C^0(U) = \text{Hom}(C_0(U), \mathbb{R}(U))$ where $C_0(U)$ is the vector-space generated by the 0-simplices $\Delta_0 \rightarrow U$ which are really points in U . Hence as $C_0(U)$ can be written as

$$\left\{ \sum_{i=0}^n r_i x_i : n \in \mathbb{Z}_{\geq 0}, r_i \in \mathbb{R}(U), x_i \in U \right\}.$$

Hence for any $f \in C^0(U)$, for any $x \in C_0(U)$ there exist $n \in \mathbb{Z}_{\geq 0}$, $r_i \in \mathbb{R}(U)$, $x_i \in U$ such that,

$$f(x) = f\left(\sum_{i=0}^n r_i x_i\right) = \sum_i r_i f(x_i).$$

So the function f is really defined by how it behaves on $U \rightarrow \mathbb{R}(U)$. So we can reinterpret $C^0(U)$ as functions $U \rightarrow \mathbb{R}(U)$. Using this information, we begin by showing that

$$0 \rightarrow \mathbb{R} \xrightarrow{\varepsilon_M} C^0 \xrightarrow{d^0} C^1$$

is exact. Note that $\psi \in \ker d^0 \iff \psi \circ \partial^1 = 0$. Now for any σ , $\partial^1(\sigma) = \sigma \circ F_{0,1} - \sigma \circ F_{1,1}$, where $F_{i,1}$ are really just points. Hence $\psi \circ \partial^1 = 0$ if, and only if $\psi(x) = \psi(y)$ whenever x and y are in the same path-component (as there is then some 1-simplex $\sigma : \Delta_1 = [0, 1] \rightarrow U$ with $\sigma(0) = x$ and $\sigma(1) = y$). So if, and only if, ψ is locally constant. So $\ker d^0$ constitutes the locally constant functions over $\mathbb{R}$. However, we note that $\varepsilon_M(U)$ is defined to be inclusion map of locally constant functions on U (so ε_M is a monomorphism). Thus giving exactness. Passing this through the sheafification functor, we conclude that

$$0 \rightarrow \mathbb{R} \hookrightarrow \mathcal{C}^0 \rightarrow \mathcal{C}^1$$

is exact. Since X is a manifold, X is locally contractible, consider for any $x \in X$ a contractible neighborhood U_p . Then,

$$C^{n-1}(U_p) \rightarrow C^n(U_p) \rightarrow C^{n+1}(U_p)$$

is exact since, $C_{n+1}(U_p) \rightarrow C_n(U_p) \rightarrow C_{n-1}(U_p)$ is exact by Corollary 5.2.7 (using that any open $U \subset X$ of a manifold X is a submanifold). So the sheafification is also. Hence we get exactness at the stalks, which is equivalent to the standard exactness, completing the proof. □

Before we are able to prove sheaves are flabby, we will need the following lemma which gives a general condition for when the sheafification map is an epimorphism,

Lemma 5.2.16. *In a paracompact Hausdorff space X , given a presheaf $\mathcal{P}$ for which the gluing axiom holds, the canonical map of the presheaf into its sheafification $\alpha : \mathcal{P} \rightarrow F\mathcal{P}$ is epimorphic (recall this essentially means $\mathcal{P}(U) \rightarrow F\mathcal{P}(U)$ is surjective for any open $U \subset X$).*

Proof. The proof is really based on the proof of a similar statement for the Alexander Spanier sheaf, Proposition 4.4 [4]. We also used some inspiration from a stackexchange post by Eric Aulder [14]. Let $s \in F\mathcal{P}(U)$.

By paracompactness, there exists a locally finite cover $\mathcal{U}$ of U with for each $U_i \in \mathcal{U}$, corresponding elements $s_i \in \mathcal{P}(U_i)$ such that $\alpha(U_i)(s_i) = \rho_{U_i}^U(s)$. We want to make use of the fact that the gluing axiom holds for $\mathcal{P}$ to make our conclusion. Since α is not necessarily injective, we need to make some choices regarding our s_i . Recall that stalkwise, the map behaves injectively, so for i, j such that U_i and U_j intersect,

$$\rho_{U_i \cap U_j}^{U_i}(s_i) = \rho_{U_i \cap U_j}^{U_j}(s_j)$$

may not necessarily hold, while we can find a neighborhood $W_{i,j}$ of some $x \in U_i \cap U_j$ such that,

$$\rho_{W_{i,j}}^{U_i}(s_i) = \rho_{W_{i,j}}^{U_j}(s_j).$$

We can pass to a refinement of $\mathcal{U}$, $\mathcal{V}$ such that $\overline{V_i} \subset U_i$ (we use this fact without proof). Since $\mathcal{V}$ is locally finite, for any $x \in X$, x intersect only $V_{x_0}, \dots, V_{x_n}$ for some $n \in \mathbb{Z}_{\geq 0}$. Now define the set A_x to be the set

$$(\cap_j V_{x_j}) \cup (\cup_{i: x \notin \overline{V_i}} \overline{V_i})^c$$

this set is open by virtue of having a closed complement, and nonempty since $\mathcal{V}$ is still a cover. By the property, $\overline{V_i} \subset U_i$ we also guarantee that only the U_{x_i} have nontrivial intersection with the complement. This allows us to conclude that for any $y \in A$, for any i such that $y \in \overline{V_i}$, $x \in \overline{V_i}$.

Additionally, $[s_{x_i}]_x = [s_{x_j}]_x$ for all $0 \leq i, j \leq n$ so there exist open neighborhoods of x , $W_{x_i,j}$ such that $\rho_{W_{x_i,j}}^{U_{x_i}}(s_{x_i}) = \rho_{W_{x_i,j}}^{U_{x_j}}(s_{x_j})$. Define $B_x := \cup_{i,j} W_{x_i,j}$. Using these, define $U(x) := A_x \cap B_x$, this is an open neighborhood of x which bears the following properties:

- i. if $y \in U(x)$ and $y \in \overline{V_i}$ then $x \in \overline{V_i}$
- ii. The s_i for which the $U_i \supset U(x)$, all coincide on $U(x)$

Using the procedure outlined, we can assign such neighborhoods $U(y)$ to all $y \in X$. Define $s^{(y)} = \rho_{U(y)}^{U_i}(s_i)$ for i such that $U_i \supset U(y)$. For all $x, y \in X$, the $s^{(x)}$ and $s^{(y)}$ coincide on $U(x) \cup U(y)$. Therefore, using the gluing property of $\mathcal{P}$, we can obtain an $\tilde{s} \in \mathcal{P}(U)$ such that $\rho_{U(x)}^U(\tilde{s}) = s^{(x)}$. Consequently, $\tilde{s}$ maps to $s \in F\mathcal{P}(U)$ via $\alpha(U)$, completing the proof. $\square$

Using this lemma, we are able to prove the next lemma,

Lemma 5.2.17. $\mathcal{C}^n(U) \cong C^n(U)/C^n(U)_0$ where $C^n(U)_0 := \{c \in C^n(U) : \text{there exists an open cover } \mathcal{U} : \rho_{U_i}^U(c) = 0 \text{ for all } U_i \in \mathcal{U}\}$.

Proof. Proof by author. We can easily show that C^n is glueable (by gluing in the obvious manner on the basis of the $C^n(U_i)$ and then extended to the entire $C^n(U)$ by injectivity of $\mathbb{R}$; that vector spaces are injective is a standard result see Dummit and Foote Example (5) in section 10.5 after Proposition 10.5.36). Since C^n is glueable and the space X paracompact and Hausdorff, $\alpha : C^n \rightarrow C^n$ is an epimorphism by the previous lemma. Now by the first isomorphism theorem for sheaves, $\mathcal{C}^n \cong C^n / \ker \alpha$. Now, for some $U \subset X$ open, $c \in \ker \alpha(U)$ if $[c]_x = 0$ for all $x \in U$, hence, if there exists an open cover $\mathcal{U}$ such that $\rho_{U_i}^U(c) = 0$ for all $U_i \in \mathcal{U}$. Additionally, if $c \in C_{\mathcal{U}}^n(X)_0$, then $c \in \ker \alpha(U)$ since then $[c]_x = 0$ for all $x \in U$. $\square$

Lemma 5.2.18. If X is a smooth manifold, the sheaves $\mathcal{C}^n$ are flabby.

Proof. This proof is based on part of the proof of Proposition 10.69 [12]. Since for $U \subset V$, $C_n(U) \subset C_n(V)$, by passing to the dual via the contravariant Hom functor, we find that $C^n(V) \rightarrow C^n(U)$ is surjective, so C^n is a flabby presheaf. To prove that C^n is flabby, we use the previous result to obtain the diagram,

$$\begin{array}{ccc} C^n(V) & \xrightarrow{\rho_U^V} & C^n(U) \\ \downarrow & & \uparrow \\ C^n(V)/C^n(V)_0 & \xrightarrow{\tilde{\rho}_U^V} & C^n(U)/C^n(U)_0 \end{array}$$

Note that $\tilde{\rho}_U^V$ is well-defined and still surjective. Well-definedness follows since for any $c \in C_{\mathcal{U}}^n(U)$ we have that $\rho_U^V(c) = c \circ \iota$ for $\iota : C_n(U) \rightarrow C_n(V)$ the canonical inclusion and hence $[c \circ \iota]_x = [c]_x$ for all $x \in U$ so that $c \circ \iota \in C_{\mathcal{U}}^n(U)$. To complete the proof we show this diagram commutes, this follows from the fact the vertical isomorphisms are sheaf isomorphisms, namely, call the isomorphism $\alpha(U) : C^n \rightarrow C^n(U)/C^n(U)_0$, then $\rho_U^V = \alpha(U)^{-1} \circ \alpha(U) \circ \rho_U^V = \alpha(U)^{-1} \circ \rho_U^V \circ \alpha(V)$. Hence the restriction maps ρ_U^V are surjective, so C^n is flabby. $\square$

So since flabby sheaves are acyclic the de Rham-Weil theorem implies the following,

Corollary 5.2.19. *The sheaf-theoretical singular cohomology $H^\cdot(\mathcal{C})$ is isomorphic to standard cohomology $H^\cdot(X; \mathbb{R})$.*

Thus we have a sort of sheaf-theoretical method for computing the singular cohomology. What remains it to show that this indeed related to singular cohomology, which will be proven in the next section.

5.2.5 Equivalence to the Classical Singular Cohomology

In this section we outline the correspondence between the sheaf-theoretical singular cohomology defined above and the classical singular cohomology defined in [2]. Note that as defined in [2], the singular cohomology is the cohomology of the sequence,

$$0 \rightarrow C^0(X) \rightarrow C^1(X) \rightarrow \dots$$

Such that $H_{\text{sing}}^k(X; \mathbb{R}) \cong \text{Hom}_{\mathbb{R}}(C_n(X), \mathbb{R}) = (C_n(X))^*$.

Theorem 5.2.20. *Sheaf-theoretical singular cohomology is isomorphic to classical singular cohomology.*

Proof. We follow the proof of theorem 10.76 [12], with the addition of the case $k = 0$ made by the author. In Lemma 5.2.17, we have proven in particular that,

$$0 \rightarrow C^n(X)_0 \rightarrow C^n(X) \rightarrow C^n(X) \rightarrow 0$$

is a short exact sequence for all $n \in \mathbb{Z}_{\geq 0}$. This gives us the short exact sequence of cochain complexes,

$$0 \rightarrow C^n(X)_0^\cdot \rightarrow C^n(X)^\cdot \rightarrow C^n(X)^\cdot \rightarrow 0.$$

By claim 3 in the proof of Theorem 4.2.1 we have proven that this induces the long exact sequence in cohomology,

[illegible]

Since $\mathcal{C}(X)_0$ is acyclic, which we prove below in Theorem , we conclude that $H^k(C(X)) \cong H^k(\mathcal{C}(X))$ for all $k \geq 1$. Where $H^k(C(X)) = H^k_{\text{sing}}(X, \mathbb{R})$ and $H^k(\mathcal{C}(X)) = H^k(X; \underline{\mathbb{R}})$. Recall also that by Proposition 3.5.2(i),

$$H^0(X; \underline{\mathbb{R}}) \cong \{f : X \rightarrow \mathbb{R} : f \text{ locally constant}\}.$$

Also, $H_{\text{sing}}^0(X, \mathbb{R}) \cong C_0(X)^*$. Now without too much effort, it can be shown that $C_0(X)^*$ is isomorphic to the path components on X . Hence $H_{\text{sing}}^0(X, \mathbb{R})$ is isomorphic to the set of functions for which each path-component is assigned a real number, which is (isomorphic to) the set of locally constant functions $f : X \rightarrow \mathbb{R}$, completing the proof. $\square$

As mentioned in the previous proof, it remains to show that $\mathcal{C}(X)_0$ is acyclic. This is proven in the next theorem. First we introduce some important properties. First the following property from singular homology theory.

Theorem 5.2.21 (Proposition 2.21 [13]). *Define $C_n^{\mathcal{U}}(X) := \{f \in C_n(M) : \text{there exists an } i \in I \text{ such that } \text{im } f \subset U_i\}$. Then the canonical inclusion $\iota : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is a chain homotopy equivalence.*

Proof. Omitted. See Proposition 2.21 in [13] for a proof. The proof is rather technical, using a process known as Barycentric subdivision. $\square$

Using the definition in the above theorem, we define $C_{\mathcal{U}}^n = (C_n^{\mathcal{U}}(X))^*$. Since $-^*$ is a contravariant functor, $C^n(X) \rightarrow C_{\mathcal{U}}^n(X)$. Defining $C_{\mathcal{U}}^n(X)_0$ to be the kernel of this map, we obtain the following result,

Lemma 5.2.22. *Let X be a smooth manifold, the $C_{\mathcal{U}}^n(X)_0$ form a direct system with the indexing set being the open covers $\mathcal{U}$, and*

$$C^n(X)_0 = \varinjlim_{\mathcal{U}} C_{\mathcal{U}}^n(X)_0.$$

Proof. See Lemma 10.73 [12]. See Appendix B for a refresher on the direct limit. $\square$

Using these two results we are able to prove the final theorem.

Theorem 5.2.23. *The cochain complex $C^\cdot(X)_0$ is acyclic*

Proof. Consider the exact sequence of cochain complexes,

$$0 \rightarrow C_{\mathcal{U}} \rightarrow C^*(X) \rightarrow C_{\mathcal{U}}(X) \rightarrow 0.$$

By claim 3 in the proof of Theorem 4.2.1, this gives us the exact sequence,

[illegible]

Recall that when proving that singular homology was homotopy invariant, in Theorem 5.2.6, we showed that chain homotopy equivalences induces isomorphic cohomologies. Hence we can conclude by theorem 5.2.21, that $H_k(C_\bullet(X)) \cong H_k(C_\bullet^{\mathcal{U}}(X))$ for all $k \in \mathbb{Z}_{\geq 0}$. So passing to the dual, we conclude that $H^k(C^\bullet(M)) \cong H^k(C_\bullet^{\mathcal{U}}(X))$ for all $k \in \mathbb{Z}_{\geq 0}$. Hence by exactness of the above sequence, we conclude that $H^k(C_\bullet^{\mathcal{U}}(X)_0) = 0$ for all $k \geq 1$, so in other words, the cochain complex $C_\bullet^{\mathcal{U}}(X)$ is exact. Now by lemma 5.2.22, and the fact that the direct limit preserves exactness, as proven in Appendix B, we conclude that

$$\varinjlim_{\mathcal{U}} C_{\mathcal{U}}(X)_0 = C(X)_0 \text{ is acyclic.}$$

☐

Using the results described in this section we obtain the following result,

Corollary 5.2.24 (The De Rham Theorem). *Let X be a manifold and $\underline{\mathbb{R}}$ be the locally constant sheaf associated to $\mathbb{R}$, then*

$$H_{dR}^{\cdot}(X) \cong H^{\cdot}(X; \mathbb{R}) \cong H_{sing}^{\cdot}(X, \mathbb{R}).$$

Proof. The first isomorphism follows from Corollary 5.1.4 and the second from Theorem 5.2.20.

This tells us that under certain conditions, De Rham and singular cohomology are isomorphic. In particular, we see that sheaf cohomology is a useful tool in the comparison of cohomological invariants. The De Rham and singular cohomology are computed in very different ways, yet essentially boil down to being the same invariant for manifolds.

5.3 Concluding Remarks

We generalized sheaf theory a bit further than necessary in this thesis. Hopefully this suggests there are many more applications. We could have taken a further step into abstraction by working with abelian categories instead of working with the concrete category of sheaves see for example [8]. Additionally, as a result of this survey of sheaf cohomology, we know how to approach different cohomology constructions. For example, as pointed out by the supervisor, group cohomology can be defined similarly using the knowledge of homological algebra developed here, see an algebra textbook such as the one by Dummit and Foote for such a construction [3]. Given more time, we would have properly proven the homotopy invariance of the sheaf cohomology, and maybe further topics such as the cohomological dimension [5]. It may also have been interesting to see the applications of cohomology in proofs of other theorems such as a generalization of the Jordan curve theorem [4]. A further study of sheaf theory may have also been interesting, such as the connection of locally constant sheaves to flat vector bundles as pointed out by the supervisor.

A Vector Bundles

Definition A.0.1. Let M be a smooth manifold. Consider a smooth manifold E together with surjective map $\pi : E \rightarrow M$ whose fibres $\pi^{-1}(p)$ are k -dimensional vector spaces over $\mathbb{K}$ for every $p \in M$ (with k being fixed). We call this pair a $\mathbb{K}$ -**vector bundle** (for field $\mathbb{K}$) **of rank k over M** if for every $p \in M$ there exists an open neighborhood U_p of p and a diffeomorphism $\Phi_{U_p} : \pi^{-1}(U_p) \rightarrow U_p \times \mathbb{K}^m$ such that:

1. $\rho_{U_p}^{U_p \times \mathbb{K}^m} \circ \Phi_{U_p} = \pi$
2. $\Phi_{U_p}|_{E_q} : \{q\} \times \mathbb{K}^m \rightarrow \mathbb{K}^m$ for every $q \in U$

We call Φ_{U_p} the **local trivialization** at p . ◇

The local trivialization property plays an important role in this definition. Namely, since around every point there exists a neighborhood U_p and a diffeomorphism between the vector spaces assigned to the points, $\coprod_{p \in U_p} E_p$ and $U_p \times \mathbb{K}^n$, the canonical identification of vector spaces to points, we know there is smoothness between the neighboring vector fields that are assigned through π . The properties of this diffeomorphism Φ are then fairly logical from this standpoint. Since, Φ restricts to π , each of the vector fields E_p is identified to p via Φ . The second property tells us that the vector field structure of each E_p is preserved via Φ . Intuitively, the vector bundle then assigns a vector field of fixed dimension k to every point on manifold M in a manner such that there is a smooth structure between neighboring vector fields.

Remark A.0.2. When (E, π) is a smooth vector bundle over M , π is in fact a submersion, this follows by the local trivialization property. Namely, let $p \in E$ then by definition there exists an open neighborhood U of $\pi(p)$ in M such that $\Phi_U : \pi_U^{-1} \rightarrow U \times \mathbb{K}^m$ is a diffeomorphism and hence a submersion. Now the projection $\rho_U^{U \times \mathbb{K}^m}$ is per definition a submersion. Thus π is a submersion as the composition of the two. ◇

Example A.0.3. 1. The trivial smooth vector bundle of M is (E, π) such that there exists a local trivialization with neighborhood M such that E is diffeomorphic to $M \times \mathbb{R}^k$. Hence generally we study vector bundles where E has some non-trivial topology and smooth structure.

2. Given an m -dimensional real valued smooth manifold M , the tangent bundle TM is a smooth vector bundle with the canonical projection map $\pi : TM \rightarrow M, (p, v) \mapsto p$. Recall the smooth structure of the tangent bundle is induced by the smooth structure on M consisting of elements (U, φ) and given by $(\pi^{-1}(U), \tilde{\varphi})$ where the $\pi^{-1}(U) \subset TM$ induce the topology on TM and the maps are given by

$$\tilde{\varphi} : \pi^{-1}(U) \rightarrow \mathbb{R}^m \times \mathbb{R}^m, v_i \frac{\partial}{\partial \varphi_i} \Big|_p \mapsto (\varphi_1(p), \dots, \varphi_m(p), v_1, \dots, v_m)$$

so that TM is a $2m$ -dimensional smooth manifold as proven in Proposition 3.18 in [2].

Next by construction each fibre of π given by $\pi^{-1}(p) = T_p M$ for any $p \in M$ is an m -dimensional real vector field. Furthermore, for a point $p \in M$ there exists a chart (U_p, φ) with $p \in U_p$, now we define

$$\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^m, v_i \frac{\partial}{\partial \varphi_i} \Big|_p \mapsto (p, v_1, \dots, v_m).$$

Composition with $\varphi \times \text{id}_{\mathbb{R}^m}$ gives $\tilde{\varphi}$, thus since $\tilde{\varphi}$ and $\varphi \times \text{id}_{\mathbb{R}^m}$ are diffeomorphisms, Φ is (as argued in Proposition 10.4 in [2]). The properties of local trivialization clearly hold. △

Another common example is the cotangent space. Since the cotangent space is just fiber-wise the dual of the tangent space, in defining the vector bundle, it is practical define some operations on tangent bundles. For defining these, the following lemma can be used,

Lemma A.0.4 (Chart lemma). *Let M be a smooth manifold and for every $p \in M$ a vector space E_p of some fixed dimension k . Define $E := \coprod_{p \in M} E_p$ and $\pi : E \rightarrow M, (x, p) \mapsto p$. Suppose also that:*

- i. $\{U_\alpha\}_{\alpha \in I}$ is an open cover of M
- ii. For all $\alpha \in I$, $\Phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k$ is bijective and for every $p \in M$, Φ_α induces an isomorphism $E_p \cong \{p\} \times \mathbb{R}^k$ of vector fields.
- iii. For all $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$ there exists exists a smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{R})$ such that $\Phi_\alpha \circ \Phi_\beta^{-1} : (v, p) \mapsto (\tau_{\alpha\beta}(p)v, p)$ is a linear isomorphism.

Then E has a unique smooth structure (inducing the topology also of course) so that (M, E, π) form a smooth $\mathbb{R}$ -vector bundle over M with rank k and $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in I}$ as the local trivializations.

Proof. Lemma 10.6 in [2]. □

This lemma is useful for, in particular, defining operations on vector bundles such as the wedge product or dualization. We will not prove these operations here, for more details see [1].

Example A.0.5 (Covector bundle). Similarly to the tangent space $(\varphi_i)_{1 \leq i \leq m}$ induces the basis of the cotangent space at a point p via the differentials $(d\varphi_i)_p$. Hence given a smooth chart (U, φ) of M , we can define the local trivialization in a similar fashion to (2) in Example A.0.3. By a similar reasoning as with the example for tangent bundles we can establish the equivalence between sections of the cotangent bundle and differential 1-forms, since a 1-form on U is a smooth map $\omega : U \ni p \mapsto \omega_p \in T_p^*M$. △

B Category theory

We briefly give some category theory definitions.

B.0.1 Pullback and Pushout

The pullback and pushout are each others duals.

Definition B.0.1 (Pullback). Let $\mathcal{C}$ be a category. Given a diagram of the form,

$$\begin{array}{ccc} & & Y \\ & & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

consisting of objects Y, Z and morphisms f, g . A tuple $(P, \iota : P \rightarrow X, j : P \rightarrow Y)$, consisting of an object P and morphisms ι, j , is said to be the pushout if for any other tuple $(Q, \alpha : Q \rightarrow X, \beta : Q \rightarrow Y)$ there exists a unique morphism $u : P \rightarrow Q$ such that,

$$\begin{array}{ccccc} Q & & & & \\ & \searrow \beta & & & \\ & & P & \xrightarrow{j} & Y \\ & \swarrow \alpha & \downarrow \iota & & \downarrow g \\ & & X & \xrightarrow{f} & Z \end{array}$$

$\exists! u$ (dotted arrow from Q to P)

commutes. ◇

Dually,

Definition B.0.2 (Pushout). Let $\mathcal{C}$ be a category. Given a diagram of the form,

$$\begin{array}{ccc} & Y & \\ & \uparrow g & \\ X & \xleftarrow{f} & Z \end{array}$$

consisting of objects Y, Z and morphisms f, g . A tuple $(P, \iota : X \rightarrow P, j : Y \rightarrow P)$, consisting of an object P and morphisms ι, j , is said to be the pushout if for any other tuple $(Q, \alpha : X \rightarrow Q, \beta : Y \rightarrow Q)$ there exists a unique morphism $u : P \rightarrow Q$ such that,

$$\begin{array}{ccccc} Q & & & & \\ & \swarrow \beta & & & \\ & & P & \xleftarrow{j} & Y \\ & \nwarrow \alpha & \uparrow \iota & & \uparrow g \\ & & X & \xleftarrow{f} & Z \end{array}$$

$\exists! u$ (dotted arrow from P to Q)

commutes. ◇

B.1 Direct Limit

For our understanding of the direct limit we made some exercises in Dummit and Foote [3]. Intuitively, a direct limit is a method to construct objects out of smaller objects. To define the direct limit, we first define what a directed set and directed system,

Definition B.1.1. • A **directed set** is a set $A \neq \emptyset$ with a reflexive and transitive binary relation $\leq$. Such that for any $a, b \in A$ there exists a $c \in A$ such that $a, b \leq c$.

- Let $(I, \leq)$ be a directed set, and $\{A_i : i \in I\}$ an indexed set of R -modules (or rings or groups). Also let $f_{i,j} : A_i \rightarrow A_j$ be homomorphisms when $i \leq j$ and such that

- $f_{i,i} = \text{id}_{A_i}$
- The composition is transitive, $f_{i,k} = f_{j,k} \circ f_{i,j}$ for all $i \leq j \leq k$.

Then we call $(A_i, f_{i,j})_{i,j \in I}$ a **directed system**. ◇

Definition B.1.2 (Direct limit of an algebraic object). Given a directed system $(A_i, f_{i,j})_{i,j \in I}$, the direct limit, is defined as,

$$\varinjlim A_i := \coprod_i A_i / \sim$$

where for $x_i \in A_i$ and $x_j \in A_j$, $x_i \sim x_j$ if, and only if, there exists a k such that $i, j \leq k$ and $f_{i,k}(x_i) = f_{j,k}(x_j)$ ◇

Example B.1.3 (Stalk). In the definition of the stalk of a sheaf, we claimed that the stalk can be expressed as a direct limit. We work this out here. Let S be a sheaf on X and $x \in X$. Let I be the open neighborhoods of x in X . This forms a directed set under the partial order of reverse inclusion $A \leq B \iff A \supset B$, which is obviously reflexive and transitive. Now in the definition of directed system, the $f_{A,B}$ represent the restriction maps ρ_B^A . Obviously, then the properties of directed system hold, as these are precisely those of the presheaf. Now,

$$\varinjlim S(U) = \coprod_{U \in \tau_X} S(U) / \sim = \{(s, U) : s \in S(U), x \in U\} / \sim.$$

Where $s \sim v$ if, and only if, there exists a W such that $W \subset W \cap V \subset U$ so such that $W \subset U \cap V$ and so that $\rho_W^U(s) = \rho_W^V(v)$, this is precisely the stalk. △

Example B.1.4 (Direct limit of rings). Let $(I, \leq)$ be a POSET (partially ordered set). For each $i \in I$ let $(A_i, +)$ be abelian. In this example, we work through exercise 7.6.8 of Dummit and Foote [3]. Suppose $(A_i, p_{i,j})_{i,j \in I}$ is a directed system and let $B = \coprod_{i \in I} A_i / \sim$. **(a) $\sim$ is an equivalence relation** Clearly reflexive, since $i \leq i$, and $p_{i,i}(a) = p_{i,i}(a)$. Now suppose $a \sim b$ then there exists a k such that $p_{i,k}(a) = p_{j,k}(b)$ so clearly $b \sim a$. Finally we show it is transitive, which is the harder part. Suppose $a \sim b$ and $b \sim c$. Then there exist a $k, k' \in I$ such that $i_a, i_b \leq k$ and $i_b, i_c \leq k'$ and $p_{i_a,k}(a) = p_{i_b,k}(b)$ and $p_{i_b,k'}(b) = p_{i_c,k'}(c)$. Since I is directed, there exists a $\tilde{k}$ such that $k, k' \leq \tilde{k}$. Thus,

$$p_{k,\tilde{k}}(p_{i_a,k}(a)) = p_{i_b,\tilde{k}}(b) = p_{k',\tilde{k}}(p_{i_b,k'}(b)) = p_{i_c,\tilde{k}}(c).$$

So $a \sim c$. **(b)** Denote by $[x]$ the class of x in A and define $p_i : A_i \rightarrow A$ by $p_i(a) = [a]$, if each $p_{i,j}$ is injective, then so is p_i . Let $[x] \in \varinjlim A_i$ be a class of $x \in A_i$ and define

$$p_i : A_i \rightarrow A, x \mapsto [x].$$

Since $p_i(x) = p_i(y) \iff x \sim y \iff$ there exists a $k \in I$ such that $p_{i,k}(x) = p_{i,k}(y)$. Now $p_{i,j}$ is injective for all $i, j \in I$ thus $x = y$. Thus p_i injective.

(c) Assume $p_{i,j}$ are group homomorphisms. Let $a \in A_i$ and $b \in A_j$. Then $[a] + [b] = [p_{i,k}(a) + p_{j,k}(b)]$ for some $k \in I$ such that $i, j \leq k$. First we prove it is well-defined. Let $a \sim a'$ and $b \sim b'$ and let $k_a, k_b \in I$ such that $k_a, k_b \leq \tilde{k}$. Then $[a] + [b] = [p_{i,\tilde{k}}(a) + p_{j,\tilde{k}}(b)] = [p_{i',\tilde{k}}(a) + p_{j',\tilde{k}}(b')] = [a'] + [b']$. Now we show this operation gives an abelian group. Well obviously the identity is $[0]$, and the addition is commutative. Given some $[a]$, $-p_{i,\tilde{k}}(a)$ is the inverse of $p_{i,\tilde{k}}(a)$ so the inverse of $[a]$ is $-[a]$. Also associativity follows by the fact that the $p_{i,j}$ are homomorphisms and by a composition rule of p .

Finally we prove the p_i in (b) are group homomorphisms. Well $p_i(x + y) = [x + y] = [p_{i,i}(x) + p_{i,i}(y)] = [x] + [y] = p_i(x) + p_i(y)$.

(d) If the A_i are all commutative rings with 1 and the $p_{i,j}$ ring homomorphisms with $1 \mapsto 1$ then A can be given the structure of a commutative ring By (c) $\varinjlim A_i$ is an abelian group and p_i are group homomorphisms. Let $a \in A_i$ and $b \in A_j$ and let $k \in I$ such that $i, j \leq k$. Define $[a] \times [b] = [p_{i,k}(a) \times p_{j,k}(b)]$. This is well-defined by similar reasoning as above using that $p_{i,j}$ are ring homomorphisms. We need to show the additional ring structure. Let $a \in A_{i_1}, b \in A_{i_2}$ and $c \in A_{i_3}$. Firstly,

$$\begin{aligned} ([a] \times [b]) \times [c] &\stackrel{\exists j}{=} [p_{i_1,j}(a) \times p_{i_2,j}(b)] \times [c] \stackrel{\exists k}{=} [p_{j,k}(p_{i_1,j}(a) \times p_{i_2,j}(b)) \times p_{i_3,k}(c)] \\ &= [p_{i_1,k}(a) \times p_{i_2,k}(b) \times p_{i_3,k}(c)] = [p_{k,k}(p_{i_1,k}(a)) \times p_{k,k}(p_{i_2,k}(b) \times p_{i_3,k}(c))] \\ &= [p_{i_1,k}(a)] \times [p_{i_2,k}(b) \times p_{i_3,k}(c)] = [a] \times ([b] \times [c]). \end{aligned}$$

Now we show it is distributive, let a, b, c be as above. By an argument of a similar nature we can conclude that $([a] + [b]) \times [c] = [a] \times [c] + [b] \times [c]$. Similar to commutativity of the addition, commutativity of the multiplication is obvious. Also the identity is $[1]$. Finally, that p_i is a ring homomorphism follows from p_i being a group homomorphism and the fact that for $a, b \in A_i$, $p_i(a \times b) = [a \times b] = [p_{i,i}(a) \times p_{i,i}(b)] = [a] \times [b] = p_i(a) \times p_i(b)$. **(e) The direct limit has the following universal property** Assume the hypothesis in (c). For all $i, j \in I$ such that $i \leq j$ the diagram,

$$\begin{array}{ccc} & C & \\ \varphi_i \nearrow & & \nwarrow \varphi_j \\ A_i & \xrightarrow{p_{i,j}} & A_j \end{array}$$

commuting (with C abelian group) implies that there exists a unique group homomorphism $\varphi : A \rightarrow C$ for all $i \in I$, the diagram,

$$\begin{array}{ccc} & C & \\ \varphi_i \nearrow & & \nwarrow \varphi \\ A_i & \xrightarrow{p_{i,j}} & A \end{array}$$

commutes.

We define $\varphi : A \rightarrow C, [a] \mapsto \varphi_i(a)$ where since $[a] \in A$, there exists an $i \in I$ such that $a \in A_i$. We show this is well-defined, let $a \sim b$ with $a \in A_i$ and $b \in A_j$. Since $a \sim b$, there exists a k such that $p_{i,k}(a) = p_{j,k}(b)$

and $a \sim p_{i,k}(a)$ hence,

$$\varphi([a]) = \varphi([p_{i,k}(a)]) = \varphi_k(p_{i,k}(a)) = \varphi_k(p_{j,k}(b)) = \varphi([p_{j,k}(b)]) = \varphi([b]).$$

Now we prove that this is a group homomorphism. Let $[a], [b] \in A$ and $a \in A_i, b \in A_j$. Then,

$$\varphi([a] + [b]) \stackrel{\exists k}{=} \varphi([p_{i,k}(a) + p_{j,k}(b)]) = \varphi_k(p_{i,k}(a) + p_{j,k}(b)) = \varphi_k(p_{i,k}(a)) + \varphi_k(p_{j,k}(b)) = \varphi([a]) + \varphi([b]).$$

So φ is indeed a group homomorphism. Finally, we show $\varphi \circ p_i = \varphi_i$ for all $i \in I$. Let $a \in A_i$ for some i , then $\varphi \circ p_i(a) = \varphi([a]) = \varphi_i(a)$. $\triangle$

We can in fact extend this to directed systems of modules using similar reasoning (as posited in exercise 10.4.25 in [3]).

Lemma B.1.5. *The direct limit preserves exactness.*

Proof. Proof by author with inspiration from stackexchange [15]. Well let $(A_i, f_{i,j})$, $(B_i, g_{i,j})$ and $(C_i, h_{i,j})$ be directed systems such that,

$$A_i \xrightarrow{\varphi_i} B_i \xrightarrow{\psi_i} C_i$$

is exact in B_i for all $i \in I$. Then we want to show that,

$$\varinjlim A_i \xrightarrow{\varphi} \varinjlim B_i \xrightarrow{\psi} \varinjlim C_i$$

is exact in $\varinjlim B_i$ where $\varphi = \varinjlim \varphi_i$ and $\psi = \varinjlim \psi_i$.

Claim 1. $\text{im } \varphi = \ker \psi$.

Let $x \in \ker \psi$ then there exists an $i \in I$ such that $x_i \in B_i$ and $g_i(x_i) = x$. So $[0] = \psi([x_i]) = [\psi_i(x_i)]$. Thus there exists a $j \geq i$ such that $h_{i,j}(\psi_i(x_i)) = 0$. Since $h_{i,j}\psi = \psi \circ g_{i,j}$, $g_{i,j}(x_i) \in \ker \psi_i$ so $g_{i,j}(x_i) \in \text{im } \varphi_i$. So there exists a $y_j \in A_j$ such that $\varphi_j(y_j) = g_{i,j}(x_i)$. Now

$$[\varphi_j(y_j)] = \varphi([y_j]) = [g_{i,j}(x_i)] = x.$$

So $x \in \text{im } \varphi$.

Now let $x \in \text{im } \varphi$. Then $x = \varphi(y)$ for some $y \in \varinjlim A_i$ so there exists an $i \in I$ such that $x = \varphi([y_i]) = [\varphi_i(y_i)]$. Now $\psi(x) = \psi([\varphi_i(y_i)]) = [\psi_i \circ \varphi_i(y_i)] = [0]$, hence $x \in \ker \psi$. So the direct limit indeed preserves exactness.

□

References

- [1] Marius Crainic. *Manifolds 2022-2023*. Sept. 2022. URL: <https://webpace.science.uu.nl/~crain101/manifolds-2022/>.
- [2] J. M Lee. *Introduction to Smooth Manifolds*. 2nd ed. Springer, 2013.
- [3] David Steven Dummit and Richard M. Foote. *Abstract algebra*. John Wiley amp; Sons, 2004.
- [4] T. A. Springer. *Sheaf theory*. 2nd ed. Utrecht: Mathematical Institute, University Utrecht, 1977.
- [5] I. Moerdijk and D.A. Pronk. *Cohomology of Sheaves*. Utrecht, 1992.
- [6] A. D. Lewis. *Chapter 1 Sheaf Theory - Queen's U*. Feb. 2014. URL: <https://mast.queensu.ca/~andrew/teaching/math942/pdf/2chapter1.pdf>.
- [7] math635. *What does it mean for a sequence of sheaves to be exact*. Dec. 2015. URL: https://math.stackexchange.com/questions/1561106/what-does-it-mean-for-a-sequence-of-sheaves-to-be-exact?noredirect=1#comment3179421_1561106.
- [8] Charles A. Weibel. *An introduction to homological algebra*. Cambridge University Press, 2011.
- [9] Robert B. Ash. *Supplement: The Long Exact Homology Sequence and Applications*. URL: <https://faculty.math.illinois.edu/~r-ash/Algebra.html>.
- [10] *Section 13.21 (015G): Cartan-Eilenberg resolutions-The Stacks project*. URL: <https://stacks.math.columbia.edu/tag/015G>.
- [11] Zarathrustra Brady. URL: <https://math.mit.edu/~notzeb/sheaf-coh.pdf>.
- [12] URL: <https://users.math.msu.edu/users/ruiterj2/Math/Documents/Fall%5C%202019/Sheaf%5C%20cohomology/Sheaf-cohomology-main.pdf>.
- [13] Allen Hatcher. *Algebraic topology*. Cambridge University Press, 2002.
- [14] Eric Auld. *When the sheafification map is a presheaf epimorphism*. June 2016. URL: <https://math.stackexchange.com/questions/1823749/when-the-sheafification-map-is-a-presheaf-epimorphism>.
- [15] Jacqueline Pauwels and Arturo Magidin. *Why do direct limits preserve exactness?* Nov. 1958. URL: <https://math.stackexchange.com/questions/121122/why-do-direct-limits-preserve-exactness>.