

Poincaré–Bendixson Theorem

Benjamin Mason

March, 2022

1 Introduction

In this essay we will consider a proof of the Poincaré–Bendixson theorem. This result was first obtained in weaker form by Henri Poincaré in 1892. Nine years later a complete proof of a more general theorem was published by Ivar Bendixson which later became known as the Poincaré–Bendixson theorem [5]. The theorem, as given in *Ordinary Differential Equations and Dynamical Systems* by Gerald Teschl [6], can be formulated as follows.

Theorem. (The Poincaré–Bendixson Theorem) *Suppose we are given a nonempty compact subset $U \subset \mathbb{R}^2$, with a continuously differentiable vector field $f \in C^1(U, \mathbb{R}^2)$ that has finitely many equilibria. If the ω –limit set of a point $y_0 \in \mathbb{R}$ is contained in U then it is either:*

1. *a fixed point*
2. *a periodic orbit*
3. *a union of finitely many equilibria $\{x_i\}$ and non-periodic orbits $\gamma(y) := \varphi_y(\mathbb{R})$ such that $\omega_{\pm}(y) \in \{x_j\}$ (where $+/-$ denotes either positive/negative time).*

Let $\emptyset \neq U \subset \mathbb{R}^2$ be a compact subset, and consider the continuously differentiable vector field $f \in C^1(U, \mathbb{R}^2)$. We will be considering the autonomous system $y' = f(y)$. We only consider the two dimensional case due to the validity of the Jordan Curve Theorem in 2 dimensions. The variant of the Jordan curve theorem we will be using is based on the following definition.

Definition 1.1 (Jordan Curve). *A **Jordan Curve** is the image of an injective continuous mapping $\varphi : S^1 \rightarrow \mathbb{R}^2$.*

Theorem 1.2 (Jordan Curve [6]). *A Jordan Curve $\mathcal{J}$ dissects $\mathbb{R}^2$ into two connected components. One of them, which we will call the interior, is bounded and the other, the exterior, is unbounded.*

Important in creating piece-wise smooth Jordan Curves are transverse segments. A transverse segment is defined as follows.

Definition 1.3. *A line segment ℓ is **transverse** to the vector field f if the vector field is not tangent to this segment and non-zero at every $y \in \ell$.*

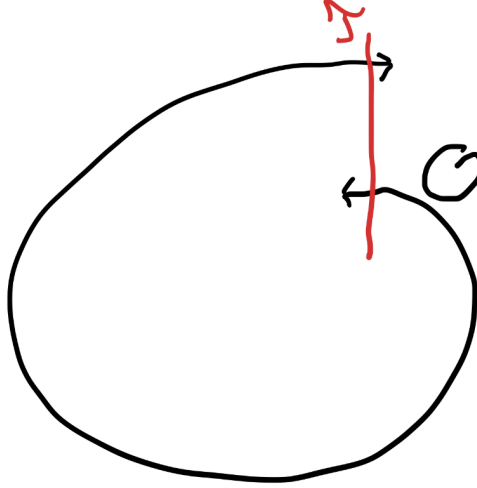


Figure 1: Example of a Jordan Curve constructed using a segment of an orbit $\mathcal{O}$ and a transverse line segment $\mathcal{T}$. Sketched by the author.

We can then construct a piece-wise differentiable Jordan curve using a transverse line segment and a segment of an orbit. We will see this again later on in the proof of the monotonicity lemma. Now we introduce ω -limit sets.

2 ω -Limit Sets

In the following sections we use ideas from Teschl [6], notes from a lecture series given at Brigham Young University in 1999 [1], notes from a lecture series given at University Utrecht [3] and a summarized proof [2]. From now on $y_0 \in \mathbb{R}^2$ refers to the initial conditions of the solution and we will refer to the solution using the notation of autonomous flows. Also we assume that $\varphi_t(y_0)$ is well defined for all $t \in \mathbb{R}$ and denote the orbit by $\gamma(y_0) := \varphi_{\mathbb{R}}(y_0)$. We begin by defining the ω -limit set.

Definition 2.1. The ω -**limit set** of an orbit $\gamma(y_0)$ is defined as $\omega_{\pm}(y_0) = \bigcap_{x \geq 0} \overline{\{\varphi_t(y_0) : t \geq \pm x\}}$.

An ω -limit set is also referred to as a limit set. In the above definition we define the ω -limit sets in both positive and negative times. In the proofs we generally only consider ω_+ under the assumption a similar argument can be made for ω_- . An ω -limit point is defined as follows,

Definition 2.2 (ω -limit point). An ω -**limit point** is a point $y \in \mathbb{R}^2$ such that there exists an increasing sequence of points $\{t_i\}_{i \in I}$ such that $\lim_{i \rightarrow \infty} t_i = \infty$ and $\lim_{i \rightarrow \infty} \varphi_{t_i}(y_0) = y$. We say that y is an ω -limit point of $y_0 \in \mathbb{R}^2$ if $y \in \omega(y_0)$.

Notice how there is a similar definition for a divergent decreasing sequence of points $\{t_i\}$. We can find a relation between these two definitions with the following important Lemma.

Lemma 2.3. The ω -limit set in a point $y_0 \in \mathbb{R}^2$ is equal to the set of all ω -limit points of y_0 .

Proof. We want to show that the set of all ω -limit points in a point y_0 is equal to $\omega_{\pm}(y_0)$. Without loss of generality we consider $\omega_+(y_0)$.

$\subset$: Consider a point $p \in \omega_+(y_0)$. Per definition then,

$$\forall x \geq 0 \forall \epsilon > 0 \exists t \geq x : d(\varphi_t(y_0), p) < \epsilon.$$

Hence for every $n \in \mathbb{N}$ there exists a $t_n \geq n$ such that $d(\varphi_{t_n}(y_0), p) < \frac{1}{n}$. Hence we have a sequence $\{t_n\}$ with points $x_n := \varphi_{t_n}(y_0)$ such that $\lim_{n \rightarrow \infty} t_n = \infty$ and $\lim_{n \rightarrow \infty} \varphi_{t_n}(y_0) = p$. Hence p is an

ω -limit point in y_0 .

$\supset$: Suppose p is an ω -limit point in y_0 . This is equivalent to saying,

$$\forall \epsilon > 0 \exists N : i \geq N \implies d(\varphi_{x_i}(y_0), p) < \epsilon.$$

Now take an arbitrary $x \in \mathbb{R}^+$ then since $\{x_i\}$ is unbounded there exists an $M \in \mathbb{N}$ such that $i \geq M \implies x_i \geq x$. Then the new statement is

$$\forall x > 0 \exists M : \forall \epsilon > 0 \exists N \geq M : i \geq N \implies d(\varphi_{x_i}(y_0), p) < \epsilon.$$

This is equivalent to saying that, $p \in \overline{\{\varphi_t(y_0) : t \geq x\}}$. Since we considered all $x \geq 0$ we find that $p \in \omega_+(y_0)$ completing the proof. $\square$

Now we prove some important properties of ω -limit sets used in this essay.

3 Some properties of ω -Sets

Associated with this set are some important properties. Consider an orbit $\gamma^+(y_0) := \varphi_{\{t \geq 0\}}(y_0)$, when this orbit is contained in a compact subset it has the following properties.

Lemma 3.1. *Suppose $\gamma^+(y_0)$ is contained in some compact subset $U \subset \mathbb{R}^2$, then:*

1. $\omega_+(y_0)$ is nonempty
2. $\omega_+(y_0)$ is closed and bounded
3. $\omega_+(y_0)$ is connected

Proof.

1. Consider the sequence $y_i = \varphi_i(y_0)$. This sequence lies in a bounded region of $\mathbb{R}^2$ by Heine-Borel. Hence by the Bolzano-Weierstrass theorem has a convergent subsequence $\{y_{i(j)}\}$. We know that $\lim_{j \rightarrow \infty} y_{i(j)} \in \omega_+(y_0)$ since $\lim_{j \rightarrow \infty} i(j) = \infty$ with $\{i(j)\}$ an increasing sequence of points. Hence $\omega_+(y_0) \neq \emptyset$. Furthermore, $\omega_+(y_0)$ is a subset of a bounded set (again by Heine-Borel) and hence also bounded.
2. The ω -limit set is an infinite intersection of closed sets and hence closed.
3. Assume $\omega_+(y_0)$ is not connected. Then there exist two disjoint opens $A, B \subset \mathbb{R}^2$ such that $\omega_+(y_0) \subset A \cup B$ and $\omega_+(y_0) \cap A, \omega_+(y_0) \cap B \neq \emptyset$. Hence there are two unbounded divergent increasing sequences $\{t_n^{(1)}\}_{n \in I}, \{t_n^{(2)}\}$ such that $\varphi_{t_n^{(1)}}(y_0) \in A$ and $\varphi_{t_n^{(2)}}(y_0) \in B$ and so that $t_{n-1}^{(2)} < t_n^{(1)} < t_n^{(2)}$ for all $n \in I$. Now consider the connected curve $\varphi_{[t_n^{(1)}, t_n^{(2)}]}(y_0)$. Since it has points in both A and B and $A \cup B$ is disconnected, we have that there exists a $t_n^{(3)} \in]t_n^{(1)}, t_n^{(2)}[$ such that $\varphi_{t_n^{(3)}}(y_0) \notin A \cup B$.

Since n was arbitrary we find an unbounded increasing sequence of such points $\{t_n^{(3)}\}$. Now since this sequence is in a bounded set it must have a convergent subsequence $\{t_{n(m)}^{(3)}\}$ where of course $\lim_{m \rightarrow \infty} t_{n(m)}^{(3)} = \infty$. This implies that $y = \lim_{m \rightarrow \infty} \varphi_{t_{n(m)}^{(3)}}(y_0) \in \omega_+(y_0)$ (since $\lim_{m \rightarrow \infty} n(m) = \infty$), a contradiction. $\square$

Now we want to develop some intuition about ω -limit sets. We establish that bounded orbits converge to their ω -limit sets (we once again, without loss of generality, only consider the positive time case).

Lemma 3.2. *Let $\gamma^+(y_0)$ be bounded. Then $\varphi_t(y_0)$ converges to the set $\omega(y_0)$ for $t \rightarrow \infty$.*

Proof. Suppose to the contrary that the orbit does not converge to its ω -limit set. Then there exists an $\epsilon > 0$ for which there are infinitely many $t_i > i$ such that

$$d(\varphi_{t_i}(y_0), \omega(y_0)) > \epsilon > 0.$$

Hence we can find an unbounded sequence $\{t_i\}$ with induction and the requirement that $t_{i+1} > \max(t_i, i)$.

However we assume that $\gamma(y_0)^+$ is bounded. Hence by Bolzano-Weierstrass there exists a subsequence $\{t_{i(j)}\}$ (still unbounded) for which

$$\lim_{j \rightarrow \infty} \varphi_{t_{i(j)}}(y_0) = y.$$

Per definition of an ω -limit point and by Lemma 2.3 we find that $y \in \omega(y_0) \implies d(y, \omega(y_0)) = 0$. However by convergence of $\varphi_{t_{i(j)}}(y_0)$ for $\epsilon/2$ there exists a j' such that,

$$d(\varphi_{t_{i(j')}}(y_0), \omega(y_0)) \leq d(\varphi_{t_{i(j')}}(y_0), y) + d(y, \omega(y_0)) = d(\varphi_{t_{i(j')}}(y_0), y) < \epsilon/2$$

a contradiction since $t_{i(j')} \in \{t_i\}$. $\square$

We prove further two properties of the ω -limit set in the next Lemma.

Definition 3.3 (Transitivity). *By **transitivity** of ω -limit sets we mean it in the general sense, that is, if $p \in \omega(q)$ and $q \in \omega(y_0)$ then $p \in \omega(y_0)$, for any $p, q, y_0 \in \mathbb{R}^2$.*

Lemma 3.4. (Invariance and Transitivity) *ω -limit sets are invariant and transitive.*

Proof. Consider an arbitrary $y_0 \in \mathbb{R}^2$. Without loss of generality we consider ω_+ . Let $p \in \omega_+(y_0)$. Then p is an ω -limit point by Lemma 2.3. Hence $\exists$ a sequence $\{t_i\}$ with $\lim_{i \rightarrow \infty} t_i = \infty$ for which $\lim_{i \rightarrow \infty} \varphi_{t_i}(y_0) = p$. Now consider $q = \varphi_t(p)$ for some $t \in \mathbb{R}$. Additionally we define $\tau_i := t_i + t$. Clearly, $\lim_{i \rightarrow \infty} \tau_i = \infty$. Furthermore by the properties of the flow,

$$\lim_{i \rightarrow \infty} \varphi_{\tau_i}(y_0) = \lim_{y \rightarrow \infty} \varphi_{\varphi_{t_i}(y_0)}(t) = \lim_{i \rightarrow \infty} y(t, 0, \varphi_{t_i}(y_0)) = \varphi_t(p).$$

The last step following from the continuity of y . Hence an arbitrary point on the orbit of a randomly chosen point in the ω -limit set is an ω -limit point that by Lemma 2.3 lies in the same set. Therefore, $\omega_+(y_0)$ is invariant.

Now we prove without loss of generality that ω_+ is transitive. Let $x, y_0, z \in \mathbb{R}^2$ be any points satisfying $x \in \omega_+(y_0)$, $z \in \omega_+(x)$. We know by Lemma 2.3 that $z \in \omega_+(x)$ implies that there exists an increasing and unbounded sequence $\{t_i\} \subset \mathbb{R}$: $\lim_{i \rightarrow \infty} \varphi_{t_i}(x) = z$. Since $x \in \omega_+(y_0)$, similarly, there exists an unbounded sequence $\{t_i^{(1)}\} \subset \mathbb{R}$ such that $\lim_{i \rightarrow \infty} \varphi_{t_i^{(1)}}(y_0) = x$. Now note that,

$$\lim_{i \rightarrow \infty} \varphi_{t_i}(\varphi_{y_0}(t_i^{(1)})) = \lim_{i \rightarrow \infty} y(t_i, 0, \varphi_{y_0}(t_i^{(1)}))^{(*)}.$$

By the continuity of y we find this is equal to,

$$y(\lim_{i \rightarrow \infty} t_i, 0, x) = \lim_{i \rightarrow \infty} \varphi_{t_i}(x) = z.$$

On the other hand we can rewrite the expression $(*)$ as follows,

$$\lim_{i \rightarrow \infty} y(t_i + t_i^{(1)}, 0, y_0) = \lim_{i \rightarrow \infty} \varphi_{t_i + t_i^{(1)}}(y_0).$$

Hence $\lim_{i \rightarrow \infty} \varphi_{t_i + t_i^{(1)}}(y_0) = z$ with $\{t_i + t_i^{(1)}\}$ an unbounded and strictly increasing sequence. So by Lemma 2.3 $z \in \omega_+(y_0)$. Hence ω_+ is transitive completing the proof. $\square$

Remark 3.5. *All of the aforementioned properties of ω -limit sets can be generalized to $\mathbb{R}^n$.*

Now we look at an example of an orbit converging to its ω -limit set.

Example 3.6. Consider the begin value problem,

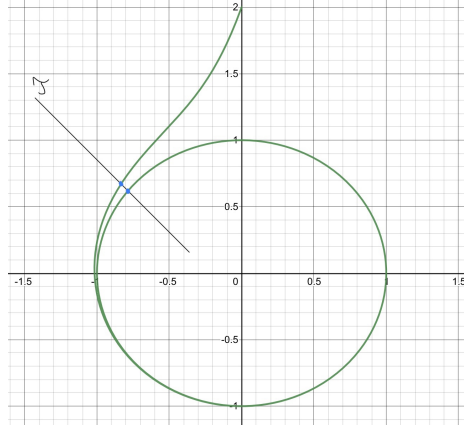
$$\begin{cases} \dot{r} = r(r^2 - 1) \\ \dot{\theta} = 1 \\ r(0) = 2, \theta(0) = 0 \end{cases}.$$

We can solve this component-wise and find through integration that,

$$\varphi_t(y_0) = \left(\frac{2}{\sqrt{4-3e^{2t}}} \right).$$

We observe that $\lim_{t \rightarrow -\infty} r(t) = 1$ and hence calculate the limit set $\omega_-(y_0)$.

Figure 2: The orbit and a transverse curve intersecting it. We see the orbit converging to its limit cycle ($\omega_-(y_0)$). Note in particular how the orbit continuously intersects the transversal $\mathcal{T}$ in a monotone fashion. Sketched by the author using desmos.



We show our conjecture that $\omega_-(y_0) = \{(r, \theta) \in \mathbb{R}^2 : r = 1\} =: A$.

$\subset$: Let $p \in \omega_-(y_0)$. By Lemma 2.3, there exists an divergent increasing unbounded sequence $\{t_i\}$ such that $\lim_{i \rightarrow \infty} t_i = -\infty$ and $\lim_{i \rightarrow \infty} \varphi_{t_i}(y_0) = p$. Component-wise we have that,

$$p = \begin{pmatrix} \lim_{i \rightarrow \infty} r(t_i) \\ \lim_{i \rightarrow \infty} t_i \end{pmatrix} = \begin{pmatrix} 1 \\ \lim_{i \rightarrow \infty} t_i \end{pmatrix}$$

since $\lim_{i \rightarrow \infty} t_i = -\infty$ and $\lim_{t \rightarrow -\infty} r(t) = 1$. Hence $p \in A$.

$\supset$: Let $x \in A$. Then $x = \begin{pmatrix} 1 \\ \theta' \end{pmatrix}$. Define a sequence $\{t_i\}$ by $t_i := -2i\pi + \theta'$. Then $\lim_{i \rightarrow \infty} t_i = -\infty$ and hence $\lim_{i \rightarrow \infty} r(t_i) = 1$. Therefore, since $\lim_{i \rightarrow \infty} -2i\pi + \theta' \mod (2\pi) = \lim_{i \rightarrow \infty} \theta' = \theta'$,

$$x = \lim_{i \rightarrow \infty} \varphi_{y_0}(t_i) = \begin{pmatrix} 1 \\ \theta' \end{pmatrix} \in \omega_-(y_0).$$

So we conclude that the ω -limit cycle of the orbit starting $(2, 0)$ is $\{(r, \theta) \in \mathbb{R}^2 : r = 1\}$ which is the unit circle in standard coordinates, this is also the behaviour we see in figure 2. Namely, the orbit seems to converge to the unit circle. The convergence to a limit cycle also clearly follows from Lemma 2.5 since the orbit $\varphi_{y_0}^-$ remains in the bounded region $\{r \leq 2\}$.

4 Some Lemmas on Monotone Sequences and Corollaries

Next we prove the Monotonicity Lemma important in obtaining some further properties of ω -sets. First we need two definitions.

Definition 4.1. A sequence of points $\{y_i\}_{i \in I}$ lying on a line segment $S \in \mathbb{R}^2$ is **monotone on S** if the sequence goes in only one direction or stays constant.

Definition 4.2. Consider a sequence of points $\{y_i\}_{i \in I}$ on an orbit $\mathcal{O} := \varphi_{\mathbb{R}}(y_0)$. The sequence of points is **monotone on $\mathcal{O}$** , if there exists an increasing sequence of points $\{t_i\}_{i \in I}$ so that $\varphi_{t_j}(y_0) = y_j$ for all $j \in I$.

Lemma 4.3 (Monotonicity Lemma). Consider a transverse line segment S and an orbit $\mathcal{O}$. If a sequence of points $\{t_i\}_{i \in I} \subset S \cap \mathcal{O}$ is monotone on the orbit $\mathcal{O}$ then it is monotone on the transverse line segment S .

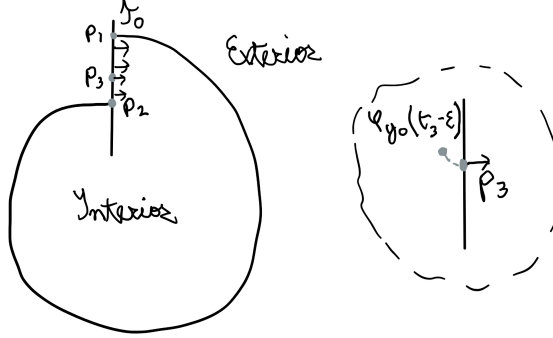


Figure 3: Proof outline monotonicity Lemma. Sketched by the author.

Proof. Consider a point $y_0 \in \mathcal{O}$ and denote the orbit $\varphi_t(y_0)$. Suppose to the contrary that this is not true. Then we can choose two points,

$$p_1, p_2, \exists t_1, t_2 : t_2 > t_1, \varphi_{t_i}(y_0) = p_i, i \in \{1, 2\}$$

with the condition that the trajectory does not intersect the transverse $\mathcal{T}$ at any point in $]t_1, t_2[$. For which there is a $t_3 > t_2$ with $p_3 \in \mathcal{T}_0$ with $\mathcal{T}_0 \subset \mathcal{T}$ the transverse segment connecting p_1 and p_2 . Now it is possible that $p_1 = p_2$ in which case we have a periodic orbit and we immediately have a monotone sequence.

Assume $p_1 \neq p_2$. Notice how $\varphi_{[t_1, t_2]}(y_0) \cup \mathcal{T}_0$ is a Jordan Curve. Hence by the Jordan curve theorem the Jordan Curve dissects $\mathbb{R}^2$ into two sets, we denote the set corresponding to the interior by $\mathcal{S}$. Notice how other orbits can only enter or leave the set $\mathcal{S}$ through the transverse line segment $\mathcal{T}_0$ (due to the uniqueness of orbits). However, since $\mathcal{T}_0$ is transverse and the vector field is continuous we can conclude, using the intermediate value theorem, that the vector field only points in or out of the set along $\mathcal{T}_0$. Hence $\mathcal{S}$ is either positive or negative invariant (meaning it can never leave $\mathcal{S}$ in either positive/negative time).

Without loss of generality, assume $\mathcal{S}$ is positively invariant. Since the vector field only points inwards along $\mathcal{T}_0$ there exists an $0 < \epsilon < t_3 - t_2$ such that $\varphi_{t_3 - \epsilon}(y_0)$ is in the exterior. However, since $t_3 - \epsilon > t_2$ and $\mathcal{S}$ is positively invariant $\varphi_{t_3 - \epsilon}(y_0) \in \mathcal{S}$, a contradiction. $\square$

Now we prove some properties of direct importance in the proof we will give of Poincaré–Bendixson theorem.

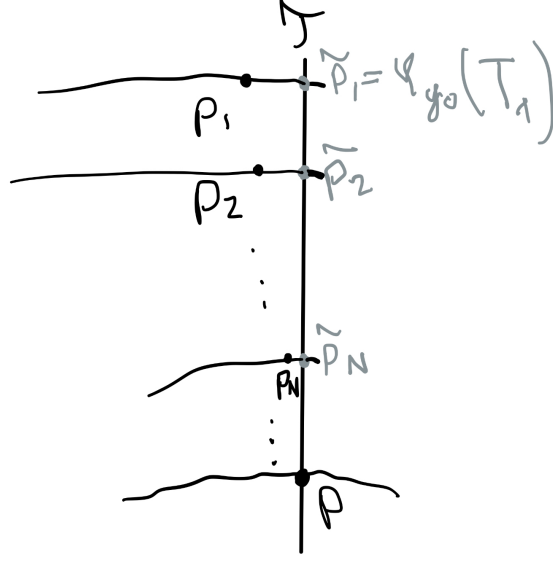


Figure 4: Lemma 4.4. Sketched by the author.

Lemma 4.4. Consider a transverse curve $\mathcal{T}$, $p \in \omega(y_0)$ and an orbit $\mathcal{O}$. Then there exists an increasing unbounded sequence $\{T_i\}$ such that $\varphi_{T_i}(y_0) \in \mathcal{T} \cap \mathcal{O}$ and $\lim_{i \rightarrow \infty} \varphi_{T_i}(y_0) = p$.

We can prove this either using the Flow-Box theorem (like the proof in [1]) or the Implicit Function Theorem (IFT). We consider the proof that uses the following 3-dimensional variant of the IFT (taken from [3]).

Theorem 4.5. Let $f \in C^1(\mathbb{R}^3, \mathbb{R})$, $(x, y) \in \mathbb{R}^3$. Consider $(x_0, y_0) : f(x_0, y_0) = 0$ and $f_y(x_0, y_0) \neq 0$. Then there exists a neighborhood $U \subset \mathbb{R}^2$ of (x_0, y_0) for which there exists a unique $g \in C^1(U, \mathbb{R})$ such that $g(y_0) = x_0$ and $f(x, g(x)) = 0$ for all $y \in \mathbb{R}^2$.

We will be using this theorem without proof. Now we prove the previous Lemma.

Proof. Firstly since $p \in \omega(y_0)$ there exists by definition an increasing unbounded sequence $\{t_i\}$ such that $\lim_{i \rightarrow \infty} \varphi_{t_i}(y_0) = p$ where we define $x_i := \varphi_{t_i}(y_0)$. Notice how t_i doesn't necessarily lie on $\mathcal{T}$. We therefore want to find a new sequence of times $\{T_i\}$ such that $\varphi_{T_i}(y_0) \in \mathcal{T}$.

Consider a point $t_i \in \mathbb{R}^2$. The transversal is a line segment, hence $\exists a \in \mathbb{R}^2$ such that a is orthogonal to every point on the transversal. Define,

$$F : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R} : (x, y) \mapsto \langle a, \varphi_x(y) - p \rangle$$

and consider a neighborhood V of p such that

$$\langle a, q - p \rangle = 0 \iff q \in \mathcal{T}.$$

This is possible when every point in the neighborhood that is on the line tangent to a lies on $\mathcal{T}$. Note how F is a C^1 function and $F(p, 0) = \langle a, p - p \rangle = 0$ and $\frac{\partial F}{\partial x}|_{(p,0)} = \langle a, f(p) \rangle \neq 0$ (since $a \neq 0$ and $f(p) \neq 0$ by definition of the transversal).

Hence by the IFT there exists a neighborhood $U \subset \mathbb{R}^2$ of p (contained in V) such that there exists a unique $\tau \in C^1(U, \mathbb{R})$ such that $F(x, \tau(x)) = 0$ for all $x \in U$ and further we have $\tau(p) = 0$.

By the continuity of τ (and hence the continuity of $\varphi_{\tau(\cdot)}(\cdot)$), there exists a neighborhood $P \subset U$ such that $\forall x \in P, \varphi_{\tau(x)}(x) \in V$. Now since $\lim_{i \rightarrow \infty} x_i = p$ there must exist an $N \in \mathbb{N}$ such that $i > N \implies x_i \in P$. Define $\tau_i := \tau(x_i)$ for $i > N$ and $\{T_i\} = \{t_{i+N} + \tau_{i+N}\}$, the result follows. $\square$

Using this result we obtain the next two desired Lemmas.

Lemma 4.6. *An ω -limit set does not intersect a transverse line segment $\mathcal{T}$ in more than one point.*

Proof. Suppose to the contrary that $\omega(y_0) \cap \mathcal{T} = \{p_1, p_2\}$. Since $p_1, p_2 \in \omega(y_0)$ using the previous Lemma we find two sequences $\{t_i^{(1)}\}$ and $\{t_i^{(2)}\}$ on $\mathcal{T}$ converging to p_1 and p_2 respectively. Using these two sequences we construct the sequence $\{t_i\}$ according to the order of the corresponding $\{t_i^{(1)}\}, \{t_i^{(2)}\}$ so that the sequence is monotone on the orbit.

Now by the Monotonicity Lemma $\{t_i\}$ is monotone on $\mathcal{T}$. This monotone sequence is in a bounded area of the transverse line segment and hence convergent. However this implies that every subsequence converges to the same point, contradicting our assumption that $p_1 \neq p_2$. $\square$

Lemma 4.7. *An ω -limit point of an ω -limit point in $\omega(y_0)$ that does not contain any equilibria lies on a periodic orbit.*

Proof. Suppose $p \in \omega(q)$ with $q \in \omega(r)$ and consider a transversal $\mathcal{T}$ that intersects p . Since $p \in \omega(q)$ there exists an increasing unbounded sequence $\{t_i\}$ for which $\lim_{i \rightarrow \infty} \varphi_q(t_i) = p$. We know by Lemma 3.1 that $p \in \omega(r)$ and hence is not a fixed point. By Lemma 4.4 there exists a sequence $\{T_i\}$ such that $x_i := \varphi_q(T_i) \in \mathcal{T}$ for all i . Since omega-limit sets are invariant and $q \in \omega(r)$ we find that $x_i \in \omega(r)$ for all i .

However, by the previous Lemma $\omega(r) \cap \mathcal{T}$ contains maximally one point. Therefore $\{x_i\}$ is a constant sequence which implies that q lies on a periodic orbit. Since $p = \lim_{i \rightarrow \infty} x_i$ we find that p lies on a periodic orbit. $\square$

We now prove one final Lemma important in proving Poincaré-Bendixson theorem.

Lemma 4.8. *If an ω -limit set $\omega(y_0)$ contains a periodic orbit $\mathcal{P}$ then $\omega(y_0) = \mathcal{P}$.*

Proof. Suppose $U := \omega(y_0) \setminus \mathcal{P} \neq \emptyset$. Since $\omega(y_0)$ is connected there exists a $y \in \mathcal{P}$ so that for all

$$\delta > 0 \exists x_\delta \in B(y, \delta) \cap U.$$

Consider a transversal $\mathcal{T}$ containing y . By a similar reasoning as in Lemma 4.4 there exists a neighborhood $y \in V$ contained in $\omega(y_0)$ on which there exists a function $\tau \in C^1(V, \mathbb{R})$ such that for all $z \in V$

$$\varphi_z(\tau(z)) \in \mathcal{T} \cap \omega(y_0).$$

Now take a $\delta_1 > 0$ such that $x_{\delta_1} \in V$. Hence by the above

$$\varphi_{x_{\delta_1}}(\tau(x_{\delta_1})) \in \mathcal{T} \cap \omega(y_0).$$

However by Lemma 3.5 we know that $\mathcal{T} \cap \omega(y_0) = \{y\}$. Therefore, $\varphi_{x_{\delta_1}}(\tau(x_{\delta_1})) \in \mathcal{P}$. However, since $\mathcal{P}$ is a closed orbit this implies that $x_{\delta_1} \in \mathcal{P}$, a contradiction. $\square$

5 The Poincaré–Bendixson theorem

Theorem 5.1 (The Poincaré–Bendixson Theorem). *Suppose we are given a nonempty compact subset $U \subset \mathbb{R}^2$, with a continuously differentiable vector field $f \in C^1(U, \mathbb{R}^2)$ that has finitely many equilibria. If the ω -limit set of a point $y_0 \in \mathbb{R}$ is contained in U then it is either:*

1. *a fixed point*
2. *a periodic orbit*
3. *a union of finitely many equilibria $\{x_i\}$ and non-closed (heteroclinic) orbits $\gamma(y) := \varphi_{\mathbb{R}}(y)$ such that $\omega_{\pm}(y) \in \{x_j\}$ (where $+/-$ denotes either positive/negative time).*

Proof.

1. Suppose $\omega(y_0)$ contains only equilibria. It can only contain finitely many, yet $\omega(y_0)$ is connected. Hence $\omega(y_0)$ is a singleton.
2. Now let $\omega(y_0)$ contain no equilibria. Then by a previous lemma $\omega(y_0)$ contains a periodic orbit $\mathcal{P}$. Thus we have that $\omega(y_0) = \mathcal{P}$.
3. Finally consider the case that $\omega(y_0)$ contains both fixed and non-vanishing points. Let $y \in \omega_{\pm}(y_0)$ be non-vanishing. We want to show that $\omega_{\pm}(y) \subset \omega_{\pm}(y_0)$ (by transitivity) contains a single point which is also a fixed point.
 - $\omega_{\pm}(y)$ contains a fixed point since otherwise a periodic orbit $\implies \omega(y_0)$ a periodic orbit, contradiction.
 - $\omega_{\pm}(y)$ does not contain a non-vanishing point. Let $z \in \omega_{\pm}(y)$ be non-vanishing and $\mathcal{T}$ be a transversal containing z . By lemma 3.3 there exists an increasing unbounded sequence $\{T_i\}$ such that $x_i := \varphi_{T_i}(y_0) \in \mathcal{T} \cap \gamma_{\pm}(y)$ where $\gamma_{\pm}(y) = \varphi_{\{\pm t > 0\}}(y)$. Since by invariance $\gamma_{\pm}(y) \subset \omega_{\pm}(y_0)$ we find with lemma 3.5 that $\mathcal{T} \cap \gamma_{\pm}(y)$ contains maximally 1 point. Hence $x_i = z$ which implies that $\gamma_{\pm}(y)$ is a periodic orbit. Therefore by lemma 3.7 $\omega(y_0)$ is a periodic orbit, contradiction since we assumed it contains equilibria.
 - Therefore by 1. $\omega_{\pm}(y)$ is a fixed point

□

We now consider an example of the third scenario.

Example 5.2. *In this example we look at a system that has a limit cycle of the third form given in Poincare-Bendixson theorem, this example has been taken from the book by Teschl [6] that has many other examples. Consider the system,*

$$\begin{cases} \dot{x} = y + x^2 - \alpha x(y - 1 + 2x^2) \\ \dot{y} = -2(1 + y)x \end{cases}$$

There are certainly three equilibria in $(0, 0), (-1, -1), (1, -1)$. In Teschl the following sketch is provided,

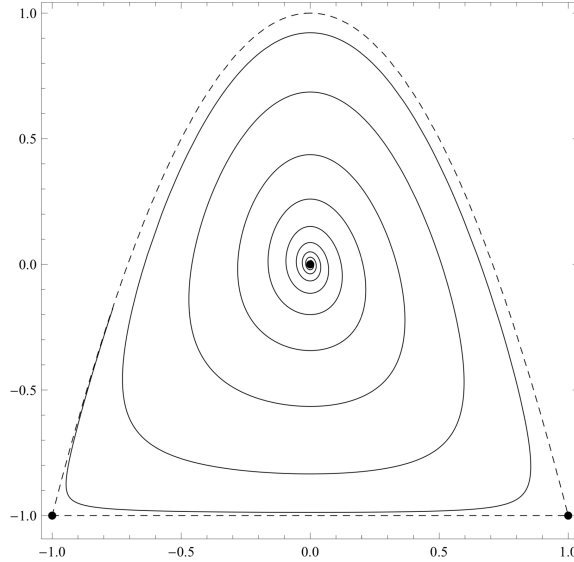


Figure 5: Above we see the third scenario described in the theorem, with the above differential equation for $\alpha = 0.1$. We have a heteroclinic cycle consisting of two equilibrium points and two orbits with one of the equilibrium points as their limit sets [6]

Note how since along the curves the vector field points in one direction and so they each have one of the equilibria as their limit sets as described by the theorem. We further see this set is invariant, it is clear why the zero solution and the two equilibrium are invariant. The invariance of the parabola can simply be shown by showing it satisfies the differential equation. We know that the parabola is given by $y = 1 - 2x^2$. Plugging this into the differential equation we find,

$$\begin{cases} \dot{x} = 1 - x^2 \\ \dot{y} = -2(2 - 2x^2)x = -4xx = \frac{d}{dt}(1 - 2x^2) \end{cases}$$

hence the parabola is a union of solutions to this differential equation and is therefore invariant.

We can work out example 2.6 much quicker with the newly found results. Further we can also prove the existence of periodic solution(s) in a system for which it would otherwise be hard to show.

Example 5.3. [4] Consider the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + y(1 - x^2 - 2y^2) \end{cases}$$

It has only one equilibrium in $(0,0)$. In polar coordinates,

$$\dot{r} = \frac{y^2}{r}(1 - r^2 - y^2).$$

For $r^2 = 1$, $\dot{r} = -y^4 \leq 0$ and for $r^2 = 1/4$, $\dot{r} = \frac{y^2}{r}(3/4 - y^2) \geq y^2 \geq 0$. Hence,

$$U := \{(r, \theta) \in \mathbb{R}^2 : 1/2 \leq r \leq 1\}$$

is a compact positively invariant set with no equilibrium. By Poincaré-Bendixson it thus contains at least one periodic orbit. Therefore proving the existence of periodic orbit(s).

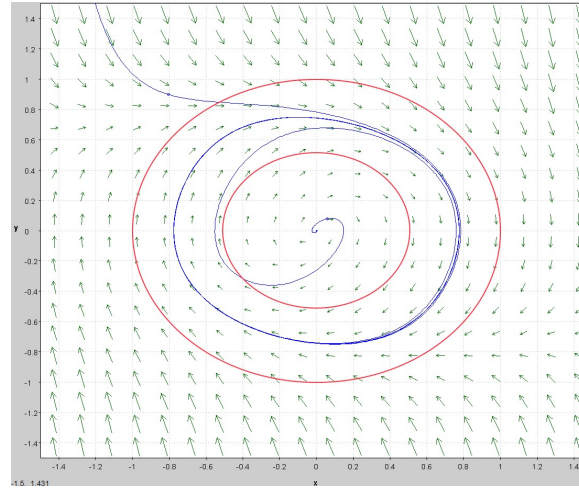


Figure 6: A sketch of the phase portrait. Here the red circles are $\{r^2 = 1\}$ and $\{r^2 = 1/4\}$ [6]. Sketched by the author using pplane9.

References

- [1] Christopher Grant. *Poincare-Bendixson Theorem*. Dec. 1999. URL: <http://www.math.byu.edu/~grant/courses/m634/f99/>.
- [2] O. Knill. *The Poincare Bendixson theorem MATH118*, O. Knill. URL: https://legacy-www.math.harvard.edu/archive/118r_spring_05/handouts/poincare.pdf.

- [3] Yuri Kuznetov. *Chapter 4. Planar ODE* - *staff.science.uu.nl*. URL: https://www.staff.science.uu.nl/~kouzn101/NLDV/Lect6_7.pdf.
- [4] J. D. Meiss. *Differential Dynamical Systems*. Society for Industrial and Applied Mathematics, 2017, pp. 222–223.
- [5] *Poincaré–Bendixson theorem*. Feb. 2022. URL: https://en.wikipedia.org/wiki/Poincar%C3%A9%E2%80%93Bendixson_theorem.
- [6] Gerald Teschl. *Ordinary Differential Equations and Dynamical Systems*. American Mathematical Society, 2020, pp. 220–227.