



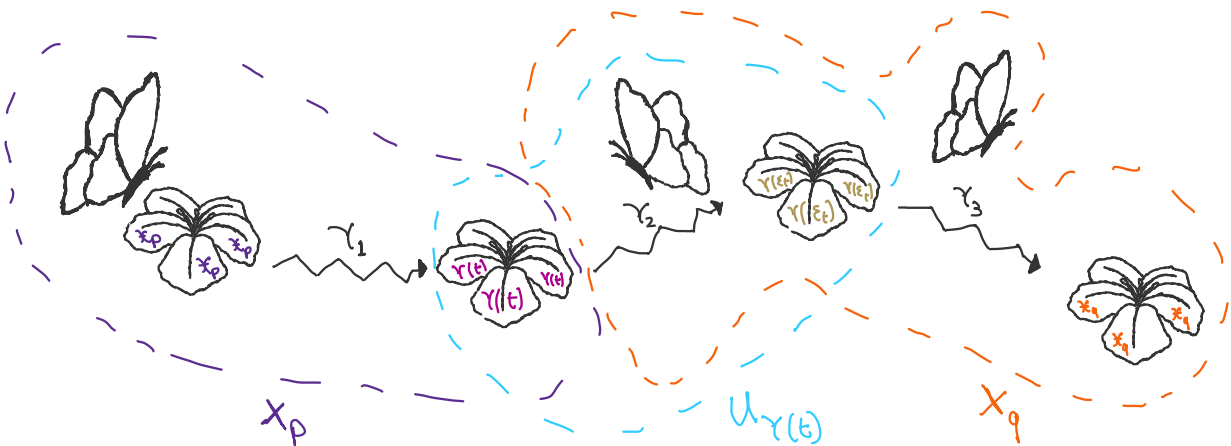
**Utrecht
University**

MASTER'S THESIS IN MATHEMATICS

Motivating and Realizing the Computation of Exodromy Representations

A Butterfly's Journey from Topological Spaces to Stratified Topological Stacks

by Ben Mason



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“Everything is going to be all right. Or, if not, everything is going to be, so let’s not worry.”

— Camille, in How to Stop Time by MATT HAIG, p. 325

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Introduction

In this thesis, we document a journey made whilst interpreting a recent result of Jansen [Jan23, Theorem 6.6], which sets up an exodromy correspondence for the moduli stack of stable curves equipped with its natural stratification. Our end goal is to describe the exodromy representation of some sheaf on the moduli stack using this correspondence. So, what is the exodromy correspondence? The short answer is that this is the monodromy correspondence for singular spaces. But this begs the question, what is the monodromy correspondence? And why do we want to tailor it to singular spaces? The monodromy correspondence is a classical result in point-set topology that identifies three separate categories:

$$\mathbf{Cov}_X \simeq \mathbf{Sh}^{\text{l.c.}}(X) \simeq \mathbf{Fun}(\Pi_1(X), \mathbf{Set})$$

which we define at the beginning of chapter 1. It originates from complex analysis where monodromy representations provide a roadmap for the analytic continuations of a holomorphic function across different paths in the space. From this perspective, sheaf theory is about descent and the local-to-global extension of holomorphic functions. As a starting point for this thesis, we introduce the modern categorical formulation of this framework, which is a justifiable leap. For example, in geometry this monodromy correspondence can be used to describe the topology of the space around singularities. In lemma 1.4.3, we see how monodromy provides an explicit framework for formally establishing the connection between Picard-Lefschetz transformations and the study of the cohomology of the family (the first higher direct image sheaf of the family).

From this perspective, exodromy is a generalization that allows us to study the topology with the singularities included. We will see how exodromy provides a framework for expressing the following statement: the first higher direct image sheaf of a family is governed by the Picard-Lefschetz theory of the family and its vanishing cycles. In order to do this, we exhibit exodromy as a generalization of monodromy to singular spaces using the formalism of [CP20; Dob23]. This requires us to define three new categories. Topological spaces will be replaced by conically stratified spaces. A P -stratified space is a topological space X along with a poset P along with a map $s_X: X \rightarrow P$ which is continuous under some natural topology associated to P . The strata of X are the fibers of s_X . Such a space is conical if moreover, locally for every $x \in X$ there is some neighborhood isomorphic to the P -stratified space $\mathbb{R}_x^d \times C(L_x)$, which serve as topological models for the singularities present in analytic geometry. Paths on such spaces are continuous maps $I \rightarrow X$ which are monotone increasing in the stratification. Such paths can be decomposed into a series of ordinary paths followed by “radial” paths, which are paths that “radiate” up the cone $C(L_x)$. This propounds the idea that exodromy should also express elements of Picard-Lefschetz theory – as these radial exit paths play the role of the local deformation retracts onto the singular fibres.

Using a notion of homotopy that is thus formulated to be decomposable into “elementary” homotopies, we then arrive at the generalized version of the fundamental groupoid, a category $\Pi_1(X, P)$ (no longer a groupoid since inter-stratum exit paths don’t invert). Representations of this category, **Set**-valued func-

tors, correspond to a suitable class of sheaves, constructible sheaves. Constructible sheaves are sheaves which are locally constant on each stratum. Under the sheaf–space correspondence on X , which establishes the equivalence between the category of local homeomorphisms over X and the sheaves on X , these sheaves correspond to stratified covering spaces also defined in the first chapter. This gives us a correspondence where once again the sheaves play a central role:

$$\underbrace{\mathbf{StratCov}_X}_{\text{A translation of the sheaf–space correspondence}} \simeq \mathbf{Sh}_P^{\text{cons}}(X) \simeq \underbrace{\mathbf{Fun}(\Pi_1(X, P), \mathbf{Set})}_{\text{A proof that relies on the careful formulation of stratified spaces}}$$

On each stratum the correspondence restricts to the monodromy correspondence, so it truly is a generalization. The proof of the more delicate side of the correspondence rests on two facts

- (a) A homotopy of exit paths can be reduced to a series of homotopies of radial exit paths.
- (b) Radial exit paths are homotopically unique in their associated principal open.

For the proof see [CP20]. After this, we interpret how to compute the exodromy representations. The decomposition into radial exit-paths and standard paths gives us one way to do this. Using this methodology, we compute the exodromy representation of the first higher direct image sheaves of a nodal family. In the second chapter, we discuss topological stacks and their categories of sheaves, in order to effectively interpret Jansen’s computation of the exit path of $\overline{\mathcal{M}}_{g,n}^{\text{top}}$. This stack arises as the solution of a moduli problem; for a motivation of moduli problems and the definition of this stack, we refer to the first section of this chapter. After motivating these, we introduce topological stacks, working with topological stacks since this is the context Jansen’s result sits in. Then in the final section we discuss stratifications for stacks, and define the categories of sheaves on topological stacks. The sheaves on the stack can be naturally described in terms of sheaves on some associated topological spaces, this is theorem 2.3.12. Using this result, we outline how the exodromy correspondence can be extended to stacks, in a way that will play a practical role in the fourth chapter. We conclude the chapter by sketching how this result also extends the proper base change theorem to topological stacks.

The third chapter of this thesis is a recount of the 1-categorical result underlying Jansen’s computation performed in [Jan23]. We define the canonical stratification on the moduli stack, the exit path category and then give the details necessary to state lemma 3.3.10.

Finally, in the fourth chapter we conjecture what this result says about the sheaf $\mathbf{R}^1\pi_*\underline{\mathbb{Q}}$ on $\overline{\mathcal{M}}_{g,n}^{\text{Top}}$ associated to the universal curve $\pi: \overline{\mathcal{C}}_{g,n}^{\text{top}} \rightarrow \overline{\mathcal{M}}_{g,n}^{\text{top}}$, using some conjectures made on the geometry of the universal curve.

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AI Disclaimer

AI was used during the drafting process, both to improve my writing and to find errors in proofs that were being drafted. It was also used to iron out/discuss ideas. Occasionally, I also used it to directly brainstorm ideas. The following results and/or proofs, listed in chronological order, were initially drafted or significantly structured with the help of AI:

Chapter 1:

1. The terminology radial exit path
2. Picture of the pinched torus and fig. 1.1
3. Terminology “cospecialization map” and “radial” exit path
4. Proof of lemma 1.3.8

Chapter 2:

5. The formulation and proof sketch of Γ -equivariant descent for sheaves.

Finally, I created the template for this thesis with the help of AI.

Chapter 1

The Exodromy Correspondence

‘Now I’m ready to start
I would rather be wrong
Than live in the shadows of your song’
🎵 *Ready to Start* — Arcade Fire

In this chapter, we aim to motivate and define exodromy. After that we cover several different perspectives of exodromy encountered during the course of this project. Most of the perspective is heavily reliant on the discussions with my supervisors, Woolf’s paper [Woo09] and Curry-Patel¹ [CP20]. The exodromy correspondence is a natural generalization of the monodromy correspondence to singular spaces.

1.1 Monodromy

The aim of this first section is to establish the monodromy correspondence:

$$\mathbf{Cov}_{/X} \simeq \mathbf{Sh}^{\text{lc.}}(X) \simeq \mathbf{Fun}(\Pi_1(X), \mathbf{Set}).$$

After doing so, we consider the sheaf of branches of the logarithm on $\mathbb{C} \setminus \{0\}$ under this correspondence, as a precursor to computing exodromy representations. Before doing this though we need to define the categories involved. We first define the fundamental groupoid on X after fixing some notation used throughout this thesis.

Notation 1.1.1. We write I for the unit interval $[0, 1]$ equipped with the Euclidean topology.

Definition 1.1.2 (Fundamental Groupoid). Let X be a topological space. The **fundamental groupoid** of X written $\Pi_1(X)$ is the category consisting of

Objects: Points $x \in X$

Morphisms: Continuous maps $\gamma: I \rightarrow X$ considered up to homotopy relative endpoints.

¹My daily supervisor pointed out to me that some errors were found in the paper by Curry and Patel. A recent bachelor’s thesis by Gieler found that not all stratified coverings considered in that paper admit the entrance path lifting property [Gie25, Theorem 4.24].

◇

Definition 1.1.3 (Local Homeomorphisms). Let X be a topological space. A continuous function $f: Y \rightarrow X$ is a **local homeomorphism** if for all $y \in Y$ there exists an open neighborhood $V \ni y$ such that $f|_V: V \rightarrow f(V)$ is a homeomorphism. The category of local homeomorphisms over X , denoted $\mathbf{LocalHomeo}_{/X}$ consists of

Objects: Local homeomorphisms $f: Y \rightarrow X$

Morphisms: Continuous functions $f: Y \rightarrow Y'$ over X .

Where composition is the usual composition over X .

◇

Next we define the locally constant sheaves on X .

Definition 1.1.4 (Locally Constant Sheaf). Let X be a topological space. A sheaf is **constant** if it can be expressed as the sheafification of a constant functor. A sheaf $\mathcal{F}$ on X is **locally constant** if there exists an open cover $\{U_i\}_I$ of X such that $\mathcal{F}|_{U_i}$ is constant. Write $\mathbf{Sh}^{\text{l.c.}}(X)$ for the full subcategory of locally constant sheaves.

◇

In sheaf theory, we have the following sheaf-space correspondence.

Theorem 1.1.5. Let X be a topological space (no assumptions necessary). The functor

$$h_{(-)/X}: \mathbf{LocalHomeo}_{/X} \rightarrow \mathbf{Sh}(X), \quad \left(Y \xrightarrow{f} X \right) \mapsto (U \mapsto \{s: U \rightarrow Y \mid f \circ s = \text{id}_U\})$$

is right adjoint to the étalé space functor

$$\text{sp}: \mathcal{F} \mapsto \left(\coprod_{U \in \text{Open}(X)} F(U) \times U \right) / \sim.$$

where the equivalence relation $\sim$ is generated by identifying a section with its restrictions. Explicitly, for any open sets $U \subset V$ and any point $x \in U$, we set:

$$(s, x) \sim (s|_U, x) \quad \text{for all } s \in \mathcal{F}(V).$$

In fact they form an adjoint equivalence

$$\mathbf{Sh}(X) \begin{array}{c} \xrightarrow{\text{sp}} \\ \cong \\ \xleftarrow{h_{(-)/X}} \end{array} \mathbf{LocalHomeo}_{/X}$$

which we call the **sheaf-space correspondence**. This equivalence restricts to a locally-constant sheaf-covering space equivalence,

$$\mathbf{Sh}^{\text{l.c.}}(X) \begin{array}{c} \xrightarrow{\text{sp}} \\ \cong \\ \xleftarrow{h_{(-)/X}} \end{array} \mathbf{Cov}_{/X}.$$

◇

Proof (Sketch). The sheaf-space correspondence is [Ten75, Chapter 2, Theorem 3.10]. The second statement follows from the fact that under the sheaf-space correspondence a covering space is a locally constant sheaf and vice versa. □

It is desirable to upgrade this correspondence further by considering an additional functor, which is defined using the unique path lifting property for covering spaces.

Definition 1.1.6. Let X be a topological space and $x_0 \in X$ a chosen basepoint. The **monodromy functor**

$$\text{Fib}_{x_0} : \mathbf{Cov}_X \longrightarrow \mathbf{Fun}(\Pi_1(X, x_0), \mathbf{Set})$$

maps a covering space $f: Y \rightarrow X$ to the $\Pi_1(X, x_0)$ **monodromy representation** defined as follows:

- **On objects:** An object $x \in X$ (viewed as an object in the fundamental groupoid $\Pi_1(X)$) maps to the fiber $f^{-1}(x)$.
- **On morphisms:** A homotopy class of paths $[\gamma]: x \rightarrow y$ in $\Pi_1(X)$ maps to the bijection

$$\rho([\gamma]): f^{-1}(x) \longrightarrow f^{-1}(y)$$

induced by unique path lifting. Specifically, $\rho([\gamma])(x')$ is the endpoint of the unique lift of γ starting at $x' \in f^{-1}(x)$. $\diamond$

If this is an equivalence, the classification of constructible sheaves reduces to determining the fundamental group and some functor category. For this to be an equivalence, however, we need to put some adjectives on our space.

Definition 1.1.7. Let X be a topological space. We say X is **path connected** (abbreviated PC) if for all $x, y \in X$ there exists a continuous function $\gamma: I \rightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Then X is **locally path connected** (LPC) if for every $x \in X$ and every neighborhood U of x there exists a path connected neighborhood $V \subseteq U$ of x .

We say that X is **locally simply connected** (LSC) if for every $x \in X$ there exists an open neighborhood $U \ni x$ such that $\pi_1(U, x) = 0$. We say that X is **semi-locally simply connected** (SLSC) if for every $x \in X$ there exists a neighborhood U of x such that every loop $\gamma: I \rightarrow U$ is contractible in X . $\diamond$

Theorem 1.1.8 (The Monodromy Correspondence). If X is a PC, LPC and SLSC topological space then the functor

$$\text{Fib}_x : \mathbf{Cov}_X \rightarrow \mathbf{Fun}(\Pi_1(X), \mathbf{Set})$$

is an equivalence.

Proof (Sketch). We essentially write over the proof sketch of [MMS24, Theorem 4.3.3]. In this case, a covering $f: Y \rightarrow X$ is locally path connected and, moreover, a disjoint union of path connected spaces [Mun00, Theorem 25.4]. Also any map $Y \rightarrow Y'$ in $\mathbf{Cov}_X$ is a covering [Mun00, Lemma 80.2(a)]. First we show that the functor Fib_x is faithful. In order to achieve this, we show that two maps

$$\begin{array}{ccc} Y' & \xrightarrow{\varphi} & Y \\ & \searrow \varphi' \swarrow & \\ & X & \end{array} \quad \begin{array}{c} f \\ f' \end{array}$$

which agree at x must agree everywhere. This is a consequence of the unique path lifting property covering spaces possess. For any $y \in Y$ let γ be any path from $f(y)$ to x . By unique path lifting, this gives a

path $\tilde{\gamma} : y \rightsquigarrow y'$ where $y' \in f^{-1}(x)$. But $\varphi \circ \tilde{\gamma}$ and $\varphi' \circ \tilde{\gamma}$ are both lifts of γ along f' , so by uniqueness they are the same path. Therefore $\varphi(y) = \varphi'(y)$.

The functor Fib_x is full by the general lifting lemma [Mun00, Lemma 79.1]. Specifically, a natural transformation between the monodromy representations $\text{Fib}_x(f)$ and $\text{Fib}_x(f')$ implies that the algebraic condition of the lifting criterion holds. This follows because the functor category of a group $\pi_1(X, x)$ into **Set** is isomorphic to the category of $\pi_1(X, x)$ -sets. Consequently, we see that a natural transformation $\eta : \text{Fib}_x(f) \Rightarrow \text{Fib}_x(f')$ is precisely a $\pi_1(X, x)$ -equivariant map between the fibers $f^{-1}(x)$ and $f'^{-1}(x)$. For a chosen pair of compatible basepoints where $y \in f^{-1}(x)$ and $y' = \eta(y)$ which is in $f'^{-1}(x)$, the equivariance of η implies that the subgroup inclusion $f_*(\pi_1(Y, y)) \subseteq f'_*(\pi_1(Y', y'))$ holds. Then, by the general lifting lemma, this corresponds to a unique morphism of covering spaces over X

$$\tilde{\eta} : Y \rightarrow Y'$$

which induces η under Fib_x , establishing fullness.

Finally, we sketch why Fib_x is essentially surjective. This, once again, rests on the fact that a $\pi_1(X, x)$ -**Set** is equivalent to a functor $\pi_1(X, x) \rightarrow \mathbf{Set}$. Every functor $F : \pi_1(X, x) \rightarrow \mathbf{Set}$ decomposes into its orbits $\cup F_i$. Note that F_i is homeomorphic to the $\pi_1(X, x)$ -**Set**

$$\pi_1(X, x) / \text{Stab}(F_i)$$

where $\text{Stab}(F_i)$ is the stabilizer and the action on this set is given by the group action on π_1 . By [Mun00, Theorem 82.1], there exists a covering $Y_i \xrightarrow{f_i} X$ such that the image of $(f_i)_*$ in $\pi_1(X, x)$ is isomorphic to H_i . Equivalently, this can be written as $F_i \xrightarrow{\sim} \text{Fib}_x(f_i)$ and consequently we obtain our desired natural isomorphism $F \xrightarrow{\sim} \text{Fib}_x(\coprod_i Y_i \rightarrow X)$. $\square$

The chosen constraints on X are sufficient but not necessary for the monodromy correspondence. Their choice is motivated by the next remark.

Remark 1.1.9. Note that, for example, non-LPC spaces for which the functor Fib_x is an equivalence do exist. An example is a modified topologist's comb with a single loop formed by the first two teeth, i.e. the subspace of $\mathbb{R}^2$ given by

$$(\{0\} \times [0, 1]) \cup \left(\bigcup_{n \in \mathbb{Z}_{>0}} \{1/n\} \times [0, 1] \right) \cup ([0, 1] \times \{0\}) \cup ([1/2, 1] \times \{1\}).$$

Consider the following topology of covers. Let $Y \rightarrow X$ be a covering space, where Y is given the **path-based covering topology** [Mun00, p. 496]. In this topology, the collection

$$U_{[\gamma]} := \{[\gamma \cdot \delta] \mid \delta \text{ is a path in } U \text{ beginning at } \gamma(1)\}$$

with indexing set

$$\{([\gamma], U) \mid \gamma(0) \text{ is a basepoint of } X \text{ and } U \text{ a path connected neighborhood of } \gamma(1)\}$$

forms a basis for the topology of Y . When we restrict our attention to LPC spaces, this path-based covering topology automatically forms a topology basis for every covering.² Consequently, the proof of

²See [Mun00, p. 496] starting at “We assert that the sets $B(U, \alpha)$ form a basis for a topology...” for a proof.

monodromy is significantly streamlined (see, for example, the proof of [Mun00, Theorem 82.1], which makes explicit use of the path-based covering topology, or the sketched proof of the monodromy correspondence where fullness of the functor Fib_X now follows from a direct application of the General Lifting Lemma). Our additional choice to restrict to PC spaces is for simplicity, as we can always decompose all categories involved in the correspondence into path-components. Then [Mun00, Corollary 82.2] necessitates the additional restriction to SLSC spaces since we want a universal covering. $\diamond$

Finally, we would like to motivate the computation of monodromy representations via the following example.

Example 1.1.10. In complex analysis the analytic continuations of the locally defined complex logarithm is closely related to the covering $\exp: \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$. In this example, we illustrate how this relation can be made explicit by means of computing an explicit representation under the monodromy correspondence. This example is inspired from this monodromy article on Wikipedia. The complex logarithm is defined to be inverse to the exponential function. Locally it can be defined as

$$\log(z) := \ln |z| + i \arg(z)$$

where the argument $\arg: \mathbb{C} \setminus \{0\} \rightarrow (a, b)$ is a bounded by $b - a < 2\pi$. The $\arg$ function is defined up to a multiple of $2\pi i$ and a choice is called a branch. The functor of branches

$$\text{Br}_{\log z}: U \mapsto \{\text{branches of } \log z \text{ on } U\}$$

is a locally constant sheaf on $\mathbb{C} \setminus \{0\}$. We could directly show that $\text{Br}_{\log z}$ corresponds to $\exp$ under the sheaf space correspondence. But, perhaps, an easier approach would be to show these both have the same monodromy representation. Observe that the path $\gamma: t \mapsto 2\pi i n t$ maps to a path $\gamma_n: t \mapsto e^{2\pi i n t}$ centered at 1 with winding number n . So the path γ_n lifts to γ . Hence, the monodromy representation associated to $\exp$ is

$$\text{Fib}_x(\exp): \pi_1(\mathbb{C} \setminus \{0\}) \rightarrow \mathbf{Set}, \quad \gamma_n \mapsto (x' \mapsto x' + 2\pi i n).$$

Once we define the parallel transport of a sheaf, later in this chapter, we verify the other half of our claim. $\diamond$

1.2 Stratified Spaces

Curry and Patel [CP20] prove exodromy for topological spaces equipped with a filtration and a conical structure modelling the singularities of objects such as algebraic varieties. In this section we define these spaces (but in the slightly more general context of poset-stratified spaces), and establish some of their properties. Our exposition is based on the article by Curry and Patel, a mini course given by van Dobben de Bruyn at Utrecht University [Dob23] and a paper by Woolf [Woo09]. We begin with the definition of a poset-stratified space.

Definition 1.2.1. Let P be a partially ordered set. The **Alexandrov topology** is the topology under which $U \subset P$ is open if it is upwards closed, that is, for all $x \in U$ if $y \geq x$ then $y \in U$. $\diamond$

Definition 1.2.2. Let P be a partially ordered set endowed with the Alexandrov topology and X a topological space. Then we say that X is **P -stratified** if it is equipped with a continuous map $s_X: X \rightarrow P$ such that each **stratum** $X_p := f^{-1}(p)$ is path connected. We will denote this space (X, P) or simply by X if the stratifying poset is clear from the context. For a point $x \in X$ we define the **stratum index of** x to be $\text{ind}_X x := f(x)$. The **star open** of the stratum X_p (or some point x_p therein) is the open neighborhood $X_{\geq p} = \cup_{q \geq p} X_q$ (is denoted $\text{st}(x_p)$). We say two strata $X_p \leq X_q$ are **adjacent** or **incident** if $\overline{X_q} \cap X_p \neq \emptyset$. $\diamond$

For example, a topological space X endowed with an n -step filtration is an $\{0, \dots, n\}$ -stratified space.

Definition 1.2.3. The map $f: X \rightarrow Y$ is a **stratified map** if for all $p \in P$ there exists a $q \in Q$ such that $f(X_p) \subseteq Y_q$ and if it is **increasing** in the following sense: if $p \leq p'$ and q, q' are the corresponding strata which f maps to, then $q \leq q'$. We denote this map by $f: (X, P) \rightarrow (Y, Q)$ and the category formed by stratified spaces and maps, **Strat**. $\diamond$

Definition 1.2.4. Let (X, P) be a stratified space. A morphism $\gamma: I \rightarrow X$ is an **exit-path** if $s_X \circ \gamma$ is monotone increasing. $\diamond$

Definition 1.2.5. Let L be a topological space. The **cone** on L is the set

$$C(L) = (L \times \mathbb{R}_{>0}) \cup \{*\}$$

equipped with the topology where $U \subseteq C(L)$ is open if and only if

- the intersection $U \cap (L \times \mathbb{R}_{>0})$ is open in $L \times \mathbb{R}_{>0}$ and
- if additionally $* \in U$ then $L \times (0, \epsilon) \subseteq U$ for some $\epsilon > 0$. $\diamond$

If we assume the space L is compact, then this definition is equivalent to the ordinary definition of the cone over L used in algebraic topology [CP20, Remark 2.5].

Definition 1.2.6. Let P be a poset. The **cone** of P , written $(P)^\triangleleft$ is P adjoined with a minimal element $*$. $\diamond$

Remark 1.2.7 (Standard Stratification on the Cone). Let (L, P) be P -stratified. Let $\mathbb{R}_{>0}$ be stratified by the 1-step filtration

$\emptyset \leq \mathbb{R}_{>0}$. The cone of L is endowed with the $(P)^\triangleleft$ -stratification

$$C(L)_p = \begin{cases} \{*\} & p = * \\ (L_p \times \mathbb{R}_{>0}) \cup \{*\} & \text{otherwise} \end{cases}$$

where $*$ is the adjoined minimal element. $\diamond$

The next definition is [CP20, Definition 2.8], but for poset stratified spaces.

Definition 1.2.8. A Hausdorff P -stratified space X is **conical** if every point $x \in X_p$ (where X_p is the stratum corresponding to $p \in P$) admits an open neighborhood $U_x \subset X$ and a $P_{>p}$ -stratified space L (called the **link** of x) such that

$$U_x \cong \mathbb{R}^d \times C(L)$$

in **Strat** for some $d \in \mathbb{Z}_{\geq 0}$. Here, $\mathbb{R}^d$ carries the trivial stratification and $C(L)$ denotes the $(P)^\triangleleft$ -stratified cone on L . We refer to these neighborhoods as **basic** or **principal** open sets. $\diamond$

Observe that, by definition, conically stratified topological spaces are locally path connected and each stratum is path connected and locally simply connected, so the monodromy correspondence holds on each stratum individually. Also, the principal opens form a basis of topology.

Lemma 1.2.9. The principal opens form a basis for the topology. A distinguished open U of x intersects all strata in the star open $\text{st}(x)$.

Proof. The first sentence is [CP20, Remark 2.11], the second sentence follows directly from the definition of the link as a $P_{>p}$ stratified space. $\square$

The ordering of the stratification is predetermined by the topology of the space.

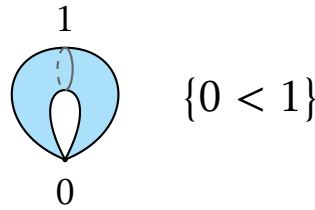
Lemma 1.2.10 (Characterization of Stratification Ordering). Let (X, P) be a conically stratified space. The ordering on the set P given by $X_p \leq X_q \iff X_q \subseteq \overline{X_p}$ coincides with the ordering of P .

Proof. Let X_p be any stratum of X . By [CP20, Remark 2.15], conically stratified spaces satisfy the **frontier condition** meaning that if $X_p \cap \overline{X_q} \neq \emptyset$ then $X_p \subseteq \overline{X_q}$. Therefore, by [WWY25, Proposition 4.1] (since a poset stratified space is in particular an Alexandrov decomposition in the terminology of the paper), the above ordering coincides with the ordering on P . $\square$

Finally, we consider two examples of singular spaces that can be endowed with a natural conical stratification.

Example 1.2.11 (Topological Manifold with Corners). [Iva90, Introduction] A **topological manifold with corners** X is a second countable Hausdorff space such that for every $x \in X$ there exists an open neighborhood $U \ni x$ homeomorphic to an open subset of $\mathbb{R}_{\geq 0}^k \times \mathbb{R}^{n-k}$. We can equip this space with a $\{0 < 1 < \dots < n\}$ -stratification where $x \in X$ belonging to the i th stratum if its open neighborhood U as above is isomorphic to $\mathbb{R}_{\geq 0}^{n-i} \times \mathbb{R}^i$. $\diamond$

Example 1.2.12 (The Pinched Torus). The pinched torus (or hollow croissant) is a $\{0 < 1\}$ -stratified space.



The singularity, i.e. the 0-stratum, has an open neighborhood isomorphic to $\mathbb{R}^0 \times C(S^1 \sqcup S^1)$ while every other point has an open neighborhood isomorphic to $\mathbb{R}^2 \times C(\emptyset)$. $\diamond$

1.2.1 The Stratified Fundamental Group

In this section we define the stratified fundamental group, following the conventions in [Dob23].

Definition 1.2.13 (Exit-path homotopy). [Dob23, p. 7] Two exit-paths $\alpha, \beta: I \rightarrow X$, from x to y are **homotopic** if there exists a stratification $I^2 \rightarrow Q$ and a morphism of conically stratified spaces $H: (I^2, Q) \rightarrow (X, P)$ such that:

(Homotopy fixing endpoints) H is a homotopy fixing endpoints from α to β .

(Homotopy in exit-paths) each $H(t, -)$ is an exit-path.

(Finite polyhedral stratification) The stratification $I^2 \rightarrow Q$ is finite and the downwards closures of strata $(I^2)_{\leq p}$ are closed polygons. $\diamond$

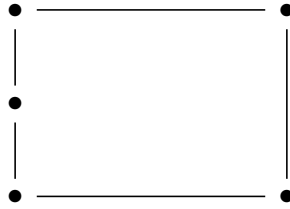
Elementary homotopies are very simple building blocks of homotopies used to simplify proofs in [CP20].

Definition 1.2.14 (Elementary Homotopy). Let Δ be the 2-simplex $[v_0, v_1, v_2]$ whose 0-strata are the edges, 1-strata the vertices and 2-stratum is the face. Let (X, P) be a conically stratified space. A stratified morphism $H: \Delta \rightarrow X$ is called an **elementary homotopy** between α and β if

- $H([v_0, v_1] \cup [v_1, v_2]) = \alpha$ and $H([v_0, v_2]) = \beta$
- $H(\Delta \setminus [v_1, v_2])$ is contained in a single stratum
- There exists a principal open U of $\alpha(0) = \beta(0)$ such that $\text{im } H \subseteq U$. $\diamond$

In the next remark we try to sketch why these are building blocks.

Remark 1.2.15. Let X be a $\mathbb{N}$ -stratified space, so a conically stratified space in the sense of [CP20]. Each elementary homotopy gives rise to an exit-path homotopy. Let $H: \Delta \rightarrow X$ be an elementary homotopy. Endowing I^2 with the stratification illustrated below



and defining a retraction $r: I^2 \rightarrow \Delta$ by collapsing the top and bottom faces ensures the composite $H \circ r$ is a valid exit-path homotopy. Moreover, elementarily related paths in the sense of [CP20, Definition 4.4] (using our notion of elementary homotopy) are homotopies in our sense. Conversely, note that the stratification on I^2 is required to be a finite polyhedral stratification. This allows us to break homotopy down into a series of elementarily related paths using reparamaterization and barycentric subdivision. $\diamond$

We define the exit-path category as follows.

Definition 1.2.16 (Exit path category). Let (X, P) be a conically stratified space. The **exit-path category** is the category $\Pi_1^{\text{Strat}}(X, P)$ consisting of

Objects: Points $x \in X$.

Morphisms: Exit-paths considered up to homotopy. $\diamond$

Lemma 1.2.17. A conically stratified space (X, P) is **locally cone-like** (recall definition 1.2.6; here we call a category with an initial object **cone-like**) in the sense that for any $x \in X$ in stratum X_p the basic open neighborhoods U of x have the property that x is the initial object of

$$\Pi_1^{\text{Strat}}(U, P_{\geq p}).$$

Proof. Let $x \in X_p$ and U be a distinguished open neighborhood of x , so there exists a $d \in \mathbb{Z}_{>0}$ such that $U \cong \mathbb{R}^d \times C(L)$. Under the isomorphism x corresponds to the point $(u, (p, r))$. For any point $y \in U$, we have a canonical path $x \rightsquigarrow y$ given by

$$\gamma(t) := \begin{cases} (u, (p, (1 - 2t)r)) & 0 \leq t \leq 1/2 \\ (2 - 2t)(u, *) + (2t - 1)y & 1/2 \leq t \leq 1 \end{cases}$$

which first pushes the point x to the apex $*$ ($0 \leq t \leq 1/2$) and then connects $(u, *)$ to y along the ray which lives in $\mathbb{R}^d \times CL \subset \mathbb{R}^n$. In [CP20, Proposition 4.11] it is shown that any entrance path $x \rightsquigarrow y$ is homotopic to γ . The construction used in their proof can be used to define a valid exit-path homotopy. After all, each elementary homotopy (in our dual sense) produces a valid exit-path homotopy. $\square$

Remark 1.2.18 (A van Kampen theorem). The van Kampen theorem for fundamental groupoids is secretly the cosheaf property (or covariant descent) of some functor. Let X be a topological space, let $A \subset X$ be a set of basepoints. For the formulation the basepoints/objects of the groupoid become important. Define $\Pi_1(X, A)$ to be the fundamental groupoid with objects A . Given two spaces $X_1, X_2 \subseteq X$ and a set of basepoints A which meets each path component of X_1, X_2 the van Kampen theorem establishes an isomorphism

$$\Pi_1(X_1 \cup X_2, A) \simeq \Pi_1(X_1, A) \coprod_{\Pi_1(X_1 \cap X_2, A)} \Pi_1(X_2, A).$$

We can define a functor,

$$\Pi_1(-): \text{Open}(X) \rightarrow \mathbf{Cat}, \quad U \mapsto \Pi_1(U, U)$$

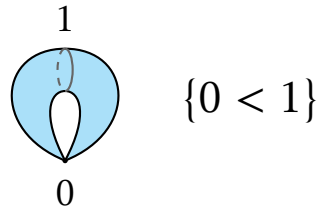
which is the pushforward on morphisms. The above discussion shows this functor preserves pushouts. In fact, this functor even satisfies the cosheaf property.

Similarly, for a conically stratified space (X, P) we can define a functor,

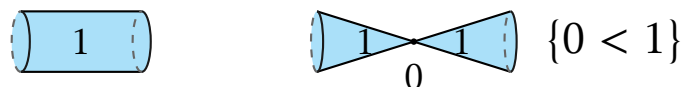
$$\Pi_1(-, P): \text{Open}(X) \rightarrow \mathbf{Cat}, \quad U \mapsto \Pi_1^{\text{Strat}}(U, P|_U)$$

which is also the pushforward on morphisms. In [CP20, Theorem 5.2] the cosheaf property is proven for entrance path categories. $\diamond$

Example 1.2.19 (Exit-path category of the pinched torus). Assuming the van Kampen theorem for exit-path categories, we can compute the stratified fundamental group of the pinched torus



We achieve this using the covering by the torus U_1 resp. the cone U_2 ,



By lemma 1.2.17, the cone U_2 has unique morphisms from the apex $*$ into the left L and right R components of the cone. Also fundamental groupoid of each of these 1-strata L, R is $\mathbb{Z}$ so we have exit-path category,

$$\mathbb{Z} \begin{array}{c} \curvearrowright \\ \circlearrowleft \end{array} L \xleftarrow{0} * \xrightarrow{0} R \begin{array}{c} \curvearrowright \\ \circlearrowleft \end{array} \mathbb{Z}$$

where in our depiction composition is given by addition in $\mathbb{Z}$. The cylinder is a $\{1\}$ -stratified space which is equivalent to an ordinary topological space. Denoting a basepoint on it by M (for the “middle” of the 1-stratum of the pinched torus), its fundamental groupoid can be written as

$$\begin{array}{c} \mathbb{Z} \\ \curvearrowright \\ M \end{array}$$

where once again composition is given by addition in $\mathbb{Z}$. Applying van Kampen results in “gluing L and R together faithfully along M ”. This produces a category,

$$* \begin{array}{c} \xrightarrow{\text{up}} \\ \xrightarrow{\text{down}} \end{array} 1 \begin{array}{c} \curvearrowright \\ \circlearrowleft \end{array} \mathbb{Z}$$

where composition on 1 should be read as addition in $\mathbb{Z}$ but we have the rules,

$$\text{up} + n = \text{up}, \quad \text{down} + n = \text{down}. \quad \diamond$$

See [Dob24] for a different approach to compute exit-path categories.

Definition 1.2.20 (Pushforward). Let $f: (Y, Q) \rightarrow (X, P)$ be a stratified map. Given an exit-path α of Y . The **pushforward** of f is a functor

$$f_*: \Pi_1^{\text{Strat}}(Y, Q) \rightarrow \Pi_1^{\text{Strat}}(X, P)$$

for which

Objects: Given an object $z \in Z$, $f_*(s) = f(z)$.

Morphisms: Given an exit-path $\alpha: z \rightsquigarrow z'$ (i.e. a map $I \rightarrow Z$) $f_*(\alpha) = [f \circ \alpha]$.

Note that $f \circ \alpha$ is again an exit-path and that if H is a homotopy of exit-paths on Y then $f \circ H$ is an exit-path homotopy on X . $\diamond$

Example 1.2.21. Let $G \curvearrowright X$ be a continuous group action on X and consider the **transformation groupoid** $G \times X \begin{array}{c} \xrightarrow{\text{pr}_2} \\ \xrightarrow{\sigma} \end{array} X$. The exit-path category on $G \times X$ is

$$\Pi_1^{\text{Strat}}(G \times X) \simeq \Pi_1^{\text{Strat}}(G) \times \Pi_1^{\text{Strat}}(X) \simeq G^{\text{disc}} \times \Pi_1^{\text{Strat}}(X)$$

where G^{disc} is the set G considered as a discrete category. The pushforwards are given by,

$$\mathrm{pr}_{2,*} : (g, \gamma) \mapsto \gamma$$

and

$$\sigma_* : (g, \gamma) \mapsto g \cdot \gamma$$

where $g \cdot \gamma$ is defined by $t \mapsto g \cdot \gamma(t)$. $\diamond$

Remark 1.2.22 (Fixing Basepoints). When convenient, we may fix a single basepoint x_p for every stratum X_p and consider the full subcategory $\Pi_{1,\mathrm{b.p.}}^{\mathrm{strat}}(X, P)$ of $\Pi_1^{\mathrm{strat}}(X, P)$ restricted to these basepoints. This does not effect our considerations because this inclusion is essentially surjective and hence an equivalence of categories: for any $y \in X_p$, we have that $x_p \cong y$ in the exit-path category (since every stratum is assumed to be path connected), implying an equivalence of categories $\Pi_{1,\mathrm{b.p.}}^{\mathrm{strat}}(X, P) \simeq \Pi_1^{\mathrm{strat}}(X, P)$. Which induces an equivalence at the level of representations:

$$\mathbf{Fun}(\Pi_1^{\mathrm{strat}}(X, P), \mathbf{Set}) \xrightarrow{\sim} \mathbf{Fun}(\Pi_{1,\mathrm{b.p.}}^{\mathrm{strat}}(X, P), \mathbf{Set}).$$

We will do this in the future without making a distinction notationally. $\diamond$

1.3 An Exodromy Correspondence for Topological Spaces

In this chapter we sketch a proof for the exodromy correspondence given in [CP20]. A slightly different variant of the proof using a lifting lemma-style argument is used in [Woo09], which would require a stratified version of the general lifting lemma [Mun00, Lemma 79.1]. To begin with we define the two other categories involved, the sheaf category and the covering category.

1.3.1 Constructible Sheaves and Stratified Coverings

Constructible Sheaves

Definition 1.3.1. Let (X, P) be a poset stratified space. A sheaf $\mathcal{F}$ is **P -constructible** if the restriction $\mathcal{F}|_{X_p}$ is a locally constant sheaf for every $p \in P$. The full subcategory of constructible sheaves is denoted $\mathbf{Sh}_P^{\mathrm{cons}}(X)$. $\diamond$

On conically stratified spaces, we have the following useful characterization of constructible sheaves.

Lemma 1.3.2. Let (X, P) be a conically stratified space and $\mathcal{F}$ a sheaf on X . Then $\mathcal{F}$ is constructible if and only if for every $x \in X$ and every pair $U \subseteq V$ of basic opens of x the restriction

$$\rho_U^V : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$$

is an isomorphism.

Proof. The dual of [CP20, Remark 3.10] implies the lemma. Every basic open U of x , restricts to a basic open on the stratum X_p associated to x . So $\mathcal{F}|_{X_p}$ is locally constant in the sense dual to [CP20, Definition 3.9]. $\square$

So stalks can be understood via their distinguished opens.

Corollary 1.3.3. Let (X, P) be conically stratified and $\mathcal{F}$ be a constructible sheaf. Then for every $U \ni x$ basic open the leg,

$$\mathcal{F}(U) \xrightarrow{\lambda_U} \mathcal{F}_x$$

is an isomorphism.

Proof. The basic open neighborhoods of x contained in U are co-initial among all open neighborhoods of x . So

$$\mathcal{F}_x \cong \operatorname{colim}_{x \in V \subset U: V \text{ basic open}} \mathcal{F}(V).$$

By lemma 1.3.2, all restrictions ρ_V^U are isomorphisms, hence the leg $\mathcal{F}(U) \rightarrow \mathcal{F}_x$ is an isomorphism. $\square$

Stratified Covering Spaces

We obtain a notion of stratified covering space and an equivalence between the constructible sheaves and the category of such spaces directly from the sheaf-space correspondence, giving us a geometric interpretation of such constructible sheaves. Let $\mathcal{F}$ be a P -constructible sheaf, then firstly $\operatorname{sp}(\mathcal{F}) \rightarrow X$ is a local homeomorphism. Moreover, since for all $p \in P$, $\mathcal{F}|_{X_p}$ is locally constant and $\operatorname{sp}(\mathcal{F})|_{X_p} = \operatorname{sp}(\mathcal{F}|_{X_p})$ the morphism $\operatorname{sp}(\mathcal{F}) \rightarrow X$ is a covering space over every stratum. We will see shortly that this characterizes the étalé spaces associated to the class of constructible sheaves. This is the dual notion to the one of branched covering spaces³ given in [CP20; Woo09].

Definition 1.3.4 (Stratified covering space). Let (X, P) and (Y, Q) be conically stratified spaces. A morphism $f: (Y, Q) \rightarrow (X, P)$ is a **stratified covering space** if

(Étale) f is a local homeomorphism

(Covering Strata) For all $p \in P$, $f|_{f^{-1}(X_p)}: f^{-1}(X_p) \rightarrow X_p$ is a covering space.

(Stratification) The strata above p are the path components of $\bigsqcup_i Y_{p_i} = f^{-1}(X_p)$. The ordering on Q is given by $p_i \leq q_j$ if and only if $Y_{q_j} \subseteq \overline{Y_{p_i}}$.

We write **StratCov** $_{/X}$ for the full subcategory **LocalHomeo** $_{/X}$ of stratified covering spaces. $\diamond$

Remark 1.3.5. The characterization of the ordering on the poset of a conically stratified space in terms of the topology, given in lemma 1.2.10, guarantees that the chosen ordering in the third/stratification axiom of a stratified covering space is the correct one. $\diamond$

So the first half of the correspondence follows quite easily from the sheaf–space correspondence.

Theorem 1.3.6 (Conically Stratified Sheaf-Space Correspondence). Let (X, P) be a conically stratified space. The sheaf–space correspondence $\mathbf{Sh}(X) \cong \mathbf{LocalHomeo}_{/X}$ restricts to an equivalence,

$$\mathbf{Sh}_P^{\operatorname{cons}}(X) \begin{array}{c} \xrightarrow{\operatorname{sp}} \\ \xleftarrow[\cong]{h_{(-)/X}} \end{array} \mathbf{StratCov}_{/X}.$$

³However, in a recent bachelors thesis Gieler shows not every branched covering space in the sense of [CP20] satisfies the entrance path lifting property [Gie25, Theorem 4.24].

Proof. We have already shown that the functor sp commutes with the restriction operation. So, given a constructible sheaf $\mathcal{F}$, the étalé space $\mathrm{sp}(\mathcal{F}) \rightarrow X$ is a stratified covering, except that we still need to show $\mathrm{sp}(\mathcal{F})$ is conical. Since f is a stratum preserving local homeomorphism, for all $x \in \mathrm{sp}(\mathcal{F})$ there exists a fundamental system of open neighborhoods $\{U_x\}$ for which $U_x \xrightarrow{\sim} f(U_x)$ is an isomorphism of stratified spaces. In other words, $\mathrm{sp}(\mathcal{F})$ has these conical opens as a topology basis, therefore it is conically stratified.

Conversely, we need to show that the sheaf of sections of a stratified covering space is a constructible sheaf. Given a stratified covering $Y \xrightarrow{f} X$, since f is continuous the sections presheaf $h_{Y/X}$ is a sheaf. Moreover, since $(h_{Y/X})|_{X_p} \cong h_{f^{-1}(X_p)/X_p}$ and $f^{-1}(X_p) \rightarrow X_p$ is a covering space, $(h_{Y/X})|_{X_p}$ is a covering space for all $p \in P$ by the equivalence between locally constant sheaves and covering spaces. So $h_{Y/X}$ is P -constructible, completing the proof. $\square$

Before formulating the second half of the correspondence we need to define parallel transport.

1.3.2 The Parallel Transport

Let (X, P) be a conically stratified space. We have a functor,

$$\rho: \mathbf{Sh}_P^{\mathrm{cons}}(X) \rightarrow \mathbf{Fun}(\Pi_1^{\mathrm{Strat}}(X, P), \mathbf{Set})$$

which assigns an **exodromy representation** $\rho_{\mathcal{F}}$ to every sheaf $\mathcal{F}$. On objects $\rho_{\mathcal{F}}$ is given by $x \mapsto \mathcal{F}_x$. The aim of this subsection is to make explicit what this functor does on morphisms.

Definition 1.3.7. Let (X, P) be a conically stratified space, $\mathcal{F}$ a P -constructible sheaf on X , and consider an exit-path $\gamma: x \rightsquigarrow y$. The **parallel transport** along γ (also called the **cospecialization map**) is a morphism

$$\rho_{\mathcal{F}}(\gamma): \mathcal{F}_x \rightarrow \mathcal{F}_y$$

induced by the functor corresponding to $\mathcal{F}$ under the exodromy correspondence (the exodromy representation of $\mathcal{F}$). $\diamond$

In order to define the parallel transport we make use of the following lemma.

Lemma 1.3.8. Let $\gamma: x \rightsquigarrow y$ be an exit-path and $\mathcal{F}$ a P -constructible sheaf on X . Then the map $\Gamma([0, 1], \gamma^* \mathcal{F}) \rightarrow \mathcal{F}_{\gamma(0)}$ is an isomorphism.

Proof. Without loss of generality, let γ be an exit-path with stratification $\{0\} \sqcup (0, 1]$. Since $\gamma^* \mathcal{F}$ is constructible with respect to this stratification, $\gamma^* \mathcal{F}$ is locally constant on $(0, 1]$. Thus constant here since $(0, 1]$ is contractible. A germ $[s]_{\gamma(0)} \in \mathcal{F}_{\gamma(0)}$ is represented by some $s \in \gamma^* \mathcal{F}([0, t))$ open neighborhood of 0 in $[0, 1]$. Its restriction $s|_{(0, t)}$ automatically extends to a compatible section $s|_{(0, 1]}$ since $\gamma^* \mathcal{F}|_{(0, 1]}$ is constant. By the sheaf property we obtain an $s' \in \gamma^* \mathcal{F}$ such that $[s']_{\gamma(0)} = [s]_{\gamma(0)}$. Since the construction of the germ is unique, the claimed isomorphism holds. $\square$

Then $\rho_{\mathcal{F}}(\gamma)$ is the map,

$$\mathcal{F}_{\gamma(0)} \rightarrow \Gamma([0, 1], \gamma^* \mathcal{F}) \rightarrow \mathcal{F}_{\gamma(1)}.$$

In words, for $x \in \mathcal{F}_{\gamma(0)}$, the parallel transport corresponds to finding a global section $s \in \gamma^* \mathcal{F}([0, 1])$ such that $[s]_0 = x$ and taking the germ $[s]_1$ at $\gamma(1)$. Parallel transport on the constant sheaf is trivial in the following sense.

Proposition 1.3.9 (Parallel transport of a constant sheaf). If $\mathcal{F}$ is the constant sheaf then for any exit-path $\gamma: x \rightsquigarrow y$, parallel transport is given by

$$\mathcal{F}_x \rightarrow \mathcal{F}_y, \quad [s]_x \mapsto [s]_y.$$

Proof. If $\mathcal{F} = \underline{A}$ is a constant then the sheaf $\gamma^* \mathcal{F}$ is constant and A -valued. In particular, for any $t \in [0, 1]$ the composite

$$A \cong \Gamma([0, 1], \gamma^* \mathcal{F}) \rightarrow \mathcal{F}_{\gamma(t)} \cong A$$

is id_A . So parallel transport is trivial. $\square$

Example 1.3.10 (Parallel Transport for $\text{Br}_{\log z}$). Now that we defined parallel transport, we determine the monodromy representation of the sheaf of branches of the logarithm,

$$\text{Br}_{\log z}: U \mapsto \{\text{branches of } \log z \text{ on } U\}$$

considered in example 1.1.10. First note that the stalk of the sheaf $\text{Br}_{\log z}$ at a point $x \in \mathbb{C} \setminus \{0\}$ is given by $\{\ln |x| + \arg x + 2\pi i k, k \in \mathbb{Z}\} \cong 2\pi i \mathbb{Z}$. Since the fundamental groupoid $\Pi_1(\mathbb{C} \setminus \{0\}) = \mathbf{B}\mathbb{Z}$ is the category with one object whose endomorphisms are the group $\mathbb{Z}$. So the functor, $\Pi_1(\mathbb{C} \setminus \{0\}) \rightarrow \mathbf{Set}$ is equivalently expressed as a homomorphism

$$\mathbb{Z} \cong \pi_1(\mathbb{C} \setminus \{0\}, 1) \rightarrow \text{Aut}\left((\text{Br}_{\log z})_x\right).$$

This automorphism is generated by $1 \in \mathbb{Z}$ which corresponds to the loop $\gamma: t \mapsto e^{2\pi i t}$. So it suffices to compute the parallel transport of γ . By definition, this corresponds to finding a global section s in $\gamma^* \text{Br}_{\log z}([0, 1])$ such that $[s]_0$ is the germ of our chosen f branch in a small neighborhood around 1. Finding such a section corresponds to producing the analytic continuation of f along γ . To this end let V_0 be a small open neighborhood of 1 and consider the branch $\log_{V_0} z = \ln |z| + i \arg(z)$ where $-\frac{\pi}{4} < \arg(z) < \frac{\pi}{4}$. Let $V_1, \dots, V_7$ be identical neighborhoods translated along γ as depicted in fig. 1.1 and define

$$\log_{V_i} = \ln |z| + i \arg(z), \quad \pi/4(i-1) < \arg(z) < \pi/4(i+1).$$

On the intersection $V_0 \cap V_7$ the identity $\log_{V_7} z = \log_{V_0} z + 2\pi i$ holds. So $\rho_{\mathcal{F}}(\gamma)(\log_{V_0} z)$ is equal to $\log_{V_0} z + 2\pi i$. Note how the argument does not depend on the choice of branch. Thus, $\rho_{\mathcal{F}}(\gamma)$ corresponds to the automorphism $f \mapsto f + 2\pi i$. We conclude that $\rho_{\mathcal{F}}$ is the homomorphism

$$n \mapsto (f \mapsto f + 2\pi i n).$$

So indeed, monodromy formalizes the relation between the sheaf of branches of the complex logarithm and the covering exp. But more importantly, the monodromy representation describes the global relation between these branches as an explicit homomorphism. $\diamond$

1.3.3 The Correspondence

In this section we prove the second half of the exodromy correspondence, which is the equivalence,

$$\mathbf{Sh}_P^{\text{cons}}(X) \simeq \mathbf{Fun}(\Pi_1^{\text{Strat}}(X, P), \mathbf{Set}).$$

Theorem 1.3.11. Let (X, P) be a conically stratified space. Then the functor ρ is an equivalence.

Proof. This should follow by an argument similar to the proof of [CP20, Theorem 6.1]. $\square$

So, the exodromy correspondence holds for conically stratified topological spaces.

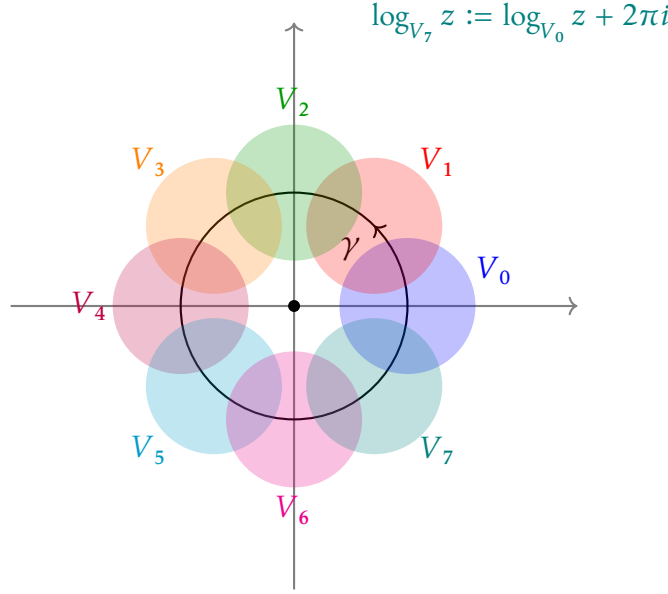


Figure 1.1: Visualization of producing an analytic extension of $\log_{V_0} z$ along $\gamma: t \mapsto e^{2\pi i t}$. Can be done with two opens rather than eight but this is more for visual aid.

1.3.4 Determining Cospecialization Maps

While the equivalence with the category of covering spaces is interesting, we will primarily be concerned with computing exodromy representations of a sheaf directly. Once the exit-path category and the stalks are computed, the primary obstacle will be computing the cospecialization maps. We define the parallel transport and prove a useful decomposition theorem. This ties in directly to our computations of exodromy representations.

A Decomposition Lemma for Exodromy Representations

In this subsection, we provide a useful characterization of exodromy representations that directs our computational approach. Essentially, in the conically stratified case, the computation of the cospecialization maps reduces to computing the monodromy representations of each stratum and expressing some sheaf restriction maps in terms of the chosen stalks. Let (X, P) be a conically stratified space and X_p, X_q be (adjacent) strata. Define the **conjugation action** by

$$\pi_1(X_p, x_p) \times \pi_1(X_q, x_q)^{\text{op}} \curvearrowright \Pi_1^{\text{Strat}}(X, P)(x_p, x_q), \quad (g, h) \cdot \gamma := g \cdot \gamma \cdot h^{-1}.$$

Note that if we have computed the monodromy of a sheaf on each stratum, it suffices to compute a cospecialization map for every orbit by functoriality. We can decompose these paths into more elementary sub-paths. To make this decomposition explicit we introduce a new definition.

Definition 1.3.12. We call an exit-path $\gamma: x \rightsquigarrow y$ **pure** or **radial** if the cospecialization $\rho_{\mathcal{F}}(\gamma)$ is isomorphic to a sheaf restriction, that is, there exist basic opens U_x of x and $U_y \subseteq U_x$ of y such that the composite

$$\mathcal{F}(U_x) \xrightarrow{\lambda_{U_x}} \mathcal{F}_x \xrightarrow{\rho_{\mathcal{F}}(\gamma)} \mathcal{F}_y \xrightarrow{\lambda_{U_y}^{-1}} \mathcal{F}(U_y)$$

is the sheaf restriction map $r_{U_y}^U$. $\diamond$

Lemma 1.3.13. Let $\gamma: x \rightsquigarrow y$ be an exit-path. If there exists a principal open U of x containing γ then γ is pure.

Proof. Since $\mathcal{F}$ is constructible, the leg $\mathcal{F}(U) \xrightarrow{\lambda_U} \mathcal{F}_{\gamma(0)}$ is an isomorphism. Consider the diagram,

$$\begin{array}{ccccc} \mathcal{F}(U) & \xrightarrow{r_{U_{\gamma(1)}}^U} & \mathcal{F}(U_{\gamma(1)}) & & \\ \downarrow & \searrow & \downarrow & & \\ \mathcal{F}_{\gamma(0)} & \longrightarrow & \Gamma([0, 1], \gamma^* \mathcal{F}) & \longrightarrow & \mathcal{F}_{\gamma(1)} \end{array}$$

where $U_{\gamma(1)} \subseteq U \cap \text{st}(\gamma(1))$ is a principal open. We detail why the triangle commutes. The colimit for $\gamma^* \mathcal{F}$ may be taken over the V that are the distinguished opens with respect to $\gamma(0)$ and that are tubular opens of γ . Indeed these are co-initial among all tubular opens. These are, in particular, also distinguished opens of $\gamma(0)$. The map $\mathcal{F}_{\gamma(0)} \rightarrow \gamma^* \mathcal{F}$ is then the morphism induced by the $\lambda_V^{-1}: \mathcal{F}_{\gamma(0)} \rightarrow \mathcal{F}(V)$, hence the triangle commutes. The square commutes since the map $\Gamma([0, 1], \gamma^* \mathcal{F}) \rightarrow \mathcal{F}_{\gamma(1)}$ is induced by restrictions of tubular opens to distinguished opens of $\gamma(1)$. Therefore the diagram commutes, so γ is pure. $\square$

A few questions remain unanswered though, for one a general exit-path is not contained in a principal open, so how do we compute its cospecialization. The next theorem decomposes general exit-paths in terms of pure exit-paths and loops.

Lemma 1.3.14 (Decomposition Lemma for Exit Paths). Let $\gamma: x_p \rightsquigarrow x_q$ be an exit-path which intersects exactly two incident strata X_p and X_q . Then γ can be dissected as

$$\gamma_3 \cdot \gamma_2 \cdot \gamma_1$$

where γ_1 and γ_3 are exit-paths in X_p respectively X_q and γ_2 is pure, see fig. 1.2. Thus the cospecialization map of γ can be expressed in terms of a pure exit-path and the monodromy on X_p and X_q .

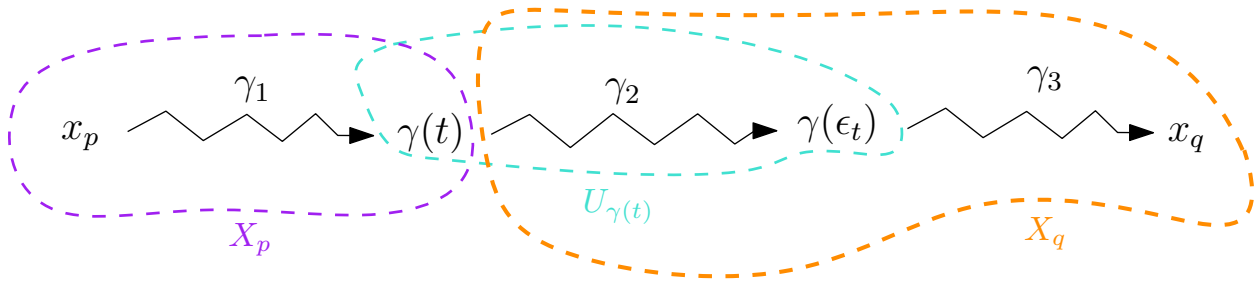


Figure 1.2: An exit-path decomposes as a pure exit-path γ_2 and paths in strata X_p and X_q written γ_1 resp. γ_3 . Compare to the cover art.

Proof. Note that $\gamma^{-1}(X_p) = [0, t]$ for some $t \in [0, 1]$. Let $U_{\gamma(t)}$ be a principal open of $\gamma(t)$. Since $U_{\gamma(t)}$ is open, there exists an $\epsilon_t > t$ such that $\gamma([t, \epsilon_t]) \subset U_{\gamma(t)}$. This shows that $\gamma|_{[t, \epsilon_t]}$ is pure. Then taking decomposition $\gamma_1 := \gamma|_{[0, t]}$, $\gamma_2 := \gamma|_{[t, \epsilon_t]}$ and $\gamma_3 := \gamma|_{[\epsilon_t, 1]}$ we get out desired decomposition, completing the proof. $\square$

Corollary 1.3.15. Let $\mathcal{F}$ be a constructible sheaf on a conically stratified space (X, P) . Denote the monodromy of $\mathcal{F}$ on the stratum X_p by $\rho_{\mathcal{F}, p}$. Let $\{\gamma_i\}$ be the orbits of the conjugation action associated to two adjacent strata X_p and X_q

$$\pi_1(X_p, x_p) \times \pi_1(X_q, x_q)^{\text{op}} \curvearrowright \Pi_1^{\text{Strat}}(X, P)(x_p, x_q).$$

Then computing the exodromy representation $\rho_{\mathcal{F}}$ is equivalent to determining

- i. the monodromy representations $\rho_{\mathcal{F}, p}$
- ii. for every x_p, x_q basepoints of adjacent strata the cospecialization $\rho_{\mathcal{F}}(r_{p,q}^i)$ of the pure exit-path $r_{p,q}^i$ associated to each orbit $\delta_{p,q}^i$ of $\Pi_1^{\text{Strat}}(X)(x_p, x_q)$. $\diamond$

1.4 Computing an Exodromy Representation

In this section we consider our working example in this thesis, the first higher direct image sheaf $\mathbf{R}^1\pi_*\mathbb{Q}$ of some proper family. Interestingly, we will see that the representation is governed by Picard-Lefschetz theory and vanishing cycles and we apply results from [CMP17, Section I.1.1] and [ACG11, pp. 146–149] to compute one such representation explicitly. Our family of interest will be the Legendre family.

Brief Detour: Defining the Legendre Family Explicitly

While this isn't strictly used later on, we choose to construct the Legendre family explicitly here for completeness. The Legendre family is the family of elliptic curves $y^2 - x(x-1)(x-t)$ over $\mathbb{P}_{\mathbb{C}}^1$. We begin by constructing this family explicitly. Firstly, by applying the Spec functor to the morphism of k -algebras

$$\mathbb{C}[x, y, \lambda]/(y^2 - x(x-1)(x-\lambda)) \rightarrow \mathbb{C}[\lambda]$$

we obtain an affine family of elliptic curves. By considering the family as the graded $\mathbb{C}[\lambda_1, \lambda_2]$ -algebra,

$$R := \mathbb{C}[\lambda_1, \lambda_2][x, y, z]/(\lambda_1 y^2 z - x(x-z)(\lambda_1 x - \lambda_2 z)).$$

the Proj construction [Har77, Exercise II.2.14] gives us a natural morphism,

$$\pi: X \rightarrow \text{Proj}(\mathbb{C}[\lambda_1, \lambda_2]) \rightarrow \mathbb{P}_{\mathbb{C}}^1, \quad f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p}).$$

Which for any open $V \subset \text{Proj } \mathbb{C}[\lambda_1, \lambda_2]$ is defined on the level of sheaves by,

$$f^{\#}(V): \mathcal{O}_{\text{Proj}(S)}(V) \rightarrow \mathcal{O}_U(f^{-1}V), \quad \left(s: V \rightarrow \coprod S_{(\mathfrak{p})}\right) \mapsto \varphi * s.$$

Where for $\mathfrak{p} \in f^{-1}(V)$ we use the notation $\varphi * s(\mathfrak{p}) := \varphi_{\mathfrak{p}}(s(f(\mathfrak{p}))) \in T_{(\mathfrak{p})}$.

This map is indeed the Legendre family since,

$$X = \{\mathfrak{p} \in \text{Proj } R: \mathfrak{p} \supsetneq \varphi(\mathbb{C}[\lambda_1, \lambda_2]_+)\} = \text{Proj } R.$$

This is the case since if $\mathfrak{p} \supset \varphi(\mathbb{C}[\lambda_1, \lambda_2]_+)$, then $\mathfrak{p} \supset R_0$ so $\mathfrak{p} = R$, which is a contradiction as $\mathfrak{p} \supsetneq R_+$. So we have constructed the Legendre family,

$$\text{Proj } \mathbb{C}[\lambda][x, y, z]/(y^2 z - (x^2 - \lambda z)(x - z)) \rightarrow \text{Proj } \mathbb{C}[\lambda].$$

It can be shown this family is proper and flat. Let $\pi: X \rightarrow D^1$ an analytic family representing the nodal degeneration. This is obtained directly from the Legendre family by taking the analytification and then taking the restriction of this family over a single node, showing that π is proper [Ray71, Proposition 3.1, Proposition 3.2]. We stratify π according to the singular fiber $D^1 \setminus \{0\} \coprod \{0\}$. Note that the sheaf $\mathbf{R}^1\pi_*\underline{\mathbb{Q}}$ is constructible with respect to this stratification in the second paragraph of the proof of [Nor02, Proposition 3.4]. Moreover, the space (D^1, P) is conically stratified, so we can compute its exodromy representation using the theory developed in this chapter. In this section we compute the exodromy representation of $\mathbf{R}^1\pi_*\underline{\mathbb{Q}}$.

1.4.1 Computing the Exit Path Category

On the open stratum, $D^1 \setminus \{0\}$ any two points are isomorphic. Fix a basepoint $x_o \in \mathbb{C} \setminus \{0\}$ then the exit-path category is equivalent to the category

$$\begin{array}{ccc} \Pi_1^{\text{Strat}}(\mathbb{C}, P)(x_o, x_o) & & \text{id}_0 \\ \curvearrowright & \xleftarrow{\Pi_1^{\text{Strat}}(\mathbb{C}, P)(0, x_o)} & \curvearrowright \\ x_o & & 0 \end{array}$$

Note that by the ordinary identification of the automorphisms of points with the fundamental group of the stratum,

$$\Pi_1^{\text{Strat}}(\mathbb{C}, P)(x_o, x_o) \cong \pi_1(\mathbb{C} \setminus \{0\}, x_o) \cong \pi_1(S^1, s_0) \cong \mathbb{Z}$$

and

$$\Pi_1^{\text{Strat}}(\mathbb{C}, P)(0, 0) \cong \pi_1(0, 0) = 0.$$

So it remains to compute the Hom-set $\Pi_1^{\text{Strat}}(0, x_o)$. But (D^1, P) is a principal open, $D^1 \cong \mathbb{R}^0 \times C(S^1)$, of 0 so this follows directly from the fact that conically stratified spaces are locally cone-like.

Remark 1.4.1 (Universal Coverings). Now that we have the exit-path category we can determine the universal coverings. It is important to note that universal coverings work with respect to a chosen basepoint. Recall that for a topological space X a universal covering, is an object that is initial in the category of pointed covering spaces. In our setting, we choose one basepoint x_p per stratum. This leads to a universal covering associated to each of these basepoints over the star stratum $\text{st}(x_p)$. The idea being, that via path-lifting, this covering classifies all paths exiting the given basepoint in the exit-path category. Indeed, the universal covering over $\text{st}(x_p)$ corresponds to the representation $\text{Hom}_{\Pi_1^{\text{Strat}}(X, P)}(x_p, -)$ (see [Dob24] for more details where it is used to compute some exit-path categories). In our example we have two universal coverings. With respect to the basepoint 0 we have universal covering,

$$\mathbb{C} \xrightarrow{\text{id}} \mathbb{C} = \text{st}(0)$$

as our computation of the exit-path category just verified and with respect to the basepoint x_o we have universal covering

$$\mathbb{C} \xrightarrow{\text{exp}} \text{st}(x_o) = \mathbb{C} \setminus \{0\}.$$

◇

1.4.2 Computing the Exodromy Representation

In this section we compute its exodromy representation. In fact, we will see this computation essentially already performed in [ACG11, pp. 146-149]. So that the exodromy representation of $\mathbf{R}^1\pi_*\underline{\mathbb{Q}}$ is essentially just a way of expressing the Picard-Lefschetz transformation and the vanishing cycle of the family π . Note firstly that by the decomposition lemma, which is lemma 1.3.14, the computation reduces to:

1. Computing the monodromy on $\{0\}$ and $\mathbb{C}\setminus\{0\}$.
2. Determining the cospecialization of a pure exit-path $0 \rightsquigarrow x_0$.

First note that the monodromy on $\{0\}$ is trivial since $\Pi_1^{\text{Strat}}(\mathbb{C}, P)(0, 0) = 0$ and

$\rho_{\mathbf{R}^1\pi_*\underline{\mathbb{Q}}, 0} : 0 \rightarrow \mathbf{Set}$ is a functor so necessarily maps to the identity. In the next lemma, we prove a slightly more general result, following through on our claim in the abstract that given a proper smooth family of curves $f : X \rightarrow S$, the monodromy of $\mathbf{R}^1f_*\underline{\mathbb{Q}}$ is determined by the Picard-Lefschetz representation of f . Before making this explicit we define what a Picard-Lefschetz representation is.

Definition 1.4.2 (Picard-Lefschetz representation). [ACG11, p. 144] Let $f : X \rightarrow S$ be a proper family of smooth n -marked genus g curves. Fix a basepoint $s_0 \in S$. A loop $\gamma : I \rightarrow S$ centered at s_0 lifts to a homeomorphism $F_{0,1}^\gamma : X_{\gamma(0)} \rightarrow X_{\gamma(1)}$ uniquely up to isotopy relative to the markings of $X_{\gamma(0)}$ so determines an element $F_{0,1}^\gamma \in \Gamma_S$ (definition 3.2.2 and definition 3.2.5). The **Picard-Lefschetz representation** is the group homomorphism,

$$\text{PL}_f : \pi_1(S, s_0) \rightarrow \Gamma_S, \quad \gamma \mapsto \overline{F_{0,1}^\gamma}. \quad \diamond$$

Let S be a Riemann surface and let $f : X \rightarrow S$ be a proper family of nodal Riemann surfaces. Stratify S according to the of singular fibers and assume there are only finitely many such fibers. Computing the exodromy of the nodal degeneration easily generalizes to this setting. So this is what we do.

Lemma 1.4.3. Let $f : X \rightarrow S$ a proper family of nodal Riemann surfaces which has an associated Picard-Lefschetz representation PL_f over the smooth locus (see for example [ACG11, Chapter IX.9 **Picard-Lefschetz revisited**] where the family is assumed to be transverse definition 4.1.1). Then the monodromy representation of $\mathbf{R}^1f_*\underline{\mathbb{Q}}$ is given by,

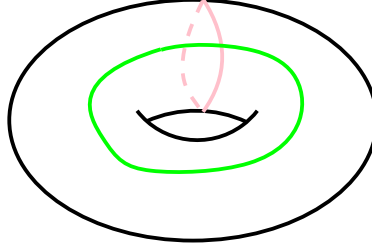
$$\rho_{\mathbf{R}^1f_*\underline{\mathbb{Q}}} : \gamma \mapsto \text{PL}_f(\gamma^{-1})^*.$$

Explicitly, for our family π , the monodromy representation on the smooth locus $\rho_{\mathbf{R}^1\pi_*\underline{\mathbb{Q}}}|_{X_{x_0}} : \mathbb{Z} \rightarrow \mathbf{Set}$ is given by

$$i \mapsto \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}.$$

Under the (c^*, d^*) -basis for cohomology where c is the curve slicing the torus into a cylinder and d is the curve which cuts the torus into an annulus (see fig. 1.3).

Proof. Let $f : X \rightarrow S$ be a proper family of smooth Riemann surfaces. The Picard-Lefschetz representation at loop γ , written $\text{PL}_f(\gamma)$ is the automorphism of X_{x_0} , considered up to isotopy, obtained by taking a trivialization $F : \gamma^*X \rightarrow X \times I$ and considering the composite,

Figure 1.3: A depiction of the curve c (red dashed line) and the curve d (green solid line).

$$X_{x_0} \xrightarrow{F_0} X_{x_0} \times \{0\} \hookrightarrow X_{x_0} \times I \xrightarrow{\text{id}_{X_{x_0}} \times \text{pr}_{\{1\}}} X_{x_0} \times I \xrightarrow{F_1^{-1}} X_{x_0}.$$

Recall that the parallel transport associated to γ is the top row of

$$\begin{array}{ccccccc} H^1(X_{x_0}; \mathbb{Q}) & \xrightarrow{\sim} & (\mathbf{R}^1 \pi_* \underline{\mathbb{Q}})_{\gamma(0)} & \xleftarrow{\sim} & \Gamma(I, \gamma^* \mathbf{R}^1 \pi_* \underline{\mathbb{Q}}) & \longrightarrow & (\mathbf{R}^1 \pi_* \underline{\mathbb{Q}})_{\gamma(1)} \xrightarrow{\sim} H^1(X_{x_0}; \mathbb{Q}) \\ & \searrow (F_0^{-1})^* & \downarrow & & \uparrow \lambda_{X_{x_0} \times I} & & \downarrow & \nearrow F_1^* \\ & & H^1(X_{x_0} \times \{0\}; \mathbb{Q}) & \xleftarrow{\sim} & H^1(X_{x_0} \times I; \mathbb{Q}) & \xrightarrow{\sim} & H^1(X_{x_0} \times \{1\}; \mathbb{Q}). \end{array}$$

The left-most triangle commutes since the map $(\mathbf{R}^1 f_* \underline{\mathbb{Q}})_{\gamma(0)} \rightarrow H^1(X_{x_0} \times \{0\})$ is defined by the collection

$$\left(H^1(\pi^{-1}(U)) \xrightarrow{\left(i_{X_{x_0}}^{-1}(U) \right)^*} H^1(X_{x_0}) \xrightarrow{\sim} H^1(X_{x_0} \times \{0\}) \right)_U$$

since stalks can be computed at the level of presheaves and $U \mapsto H^1(\pi^{-1}(U); \mathbb{Q})$ is a presheaf underlying $\mathbf{R}^1 \pi_* \underline{\mathbb{Q}}$. By a similar argument, the right-most triangle commutes, so it suffices to prove that the two squares commute. But this once again follows from the definition of our maps in the parallel transport. Consequently, by commutativity of this diagram, since tracing the bottom face yields $(\text{PL}_f(\gamma^{-1}))^*$, we conclude that

$$\rho_{\mathbf{R}^1 f_* \underline{\mathbb{Q}}}(\gamma) = (\text{PL}_f(\gamma^{-1}))^*.$$

Now to apply this result to our family π . In [ACG11, pp. 146-149] the Picard Lefschetz representation of π is proven to be of the form,

$$\text{PL}: \pi_1(\mathbb{C} \setminus \{0\}, x_0) \rightarrow \text{Aut}(X_{x_0}), \quad 1 \mapsto \delta_c$$

where δ_c is the Dehn twist associated to the curve c slicing the torus into a cylinder. Hence

$$\rho_{\mathcal{F}, x_0}(1) = (\delta_c^{-1})^*.$$

Let d be the curve cutting the torus into an annulus. Using the basis c^*, d^* for cohomology, $(\delta_c)^*$ corresponds to the claimed matrix. This exact computation is performed in [CMP17, Chapter 1.1 “The Local Monodromy Representation”], for the c_*, d_* basis on homology. But cohomology is the dual, and the matrix acting on the dual is the inverse transpose. But the inverse of the inverse transpose is the transpose, completing the proof. $\square$

Next we compute the cospecialization of a pure exit-path, once again, this is essentially already performed in [ACG11].

Definition 1.4.4 (Vanishing Cycle). Let $f: X \rightarrow S$ be a proper family of nodal Riemann surfaces and let X_s be a nodal fiber. Assume there exists an open neighborhood U of s such that X_s is the only singular fiber over U and the restriction $X|_U$ deformation retracts onto X_s . Then the **vanishing cycles** are the cycles generating the kernel of the pushforward map r_* on homology, where $r: X|_U \rightarrow X_s$ is the retraction. $\diamond$

Lemma 1.4.5. Let S be a smooth Riemann surface. Let $f: X \rightarrow S$ be a proper family of nodal Riemann surfaces such that for each nodal surface X_s there exists an open $U \ni s$ and a retract $r: U \rightarrow \{s\}$ which lifts to a retract $R: X|_U \rightarrow X_s$ such that for any smooth fiber $X_{s'}$ in $X|_U$ the vanishing cycles are a disjoint collection of embedded curves (see for example [ACG11, Lemma X.9.19]). The cospecializations of the associated sheaf $\mathbf{R}^1 f_* \mathbb{Q}$ are given by the vanishing cycles of the singularities. Explicitly, consider a pair of adjacent strata $X_p, \overline{X}_q$ with basepoints x_p respectively x_q . We can choose pure exit-path $\gamma: x_p \rightsquigarrow x_q$ (we can choose such a pure γ by the proof of the decomposition lemma, since our space is conically stratified). This path lifts to a deformation retraction $r_{p,q}: X_{x_p} \rightarrow X_{x_q}$ whose kernel vanishing cycle and the cospecialization of γ evaluates to $(r_{p,q})^*$.

Once again, consider our example of the nodal degeneration and let $\gamma: 0 \rightsquigarrow x_0$ be a pure exit-path. Then the cospecialization map is given by,

$$r_{x_0}^*: H^1(X_0; \mathbb{Q}) \rightarrow H^1(X_{x_0}; \mathbb{Q}), \quad 1 \mapsto (0, 1)$$

in the (c^*, d^*) -basis.

Proof. Let S_p, S_q be a pair of adjacent strata with basepoints s_p and s_q and $p < q$. Since S_p is the point s_p and we assumed the singular locus of the family was 0-dimensional there exists an ϵ -disc around s_p containing no other singularities U_ϵ . We restrict our attention to the family $f_\epsilon: X|_{U_\epsilon} \rightarrow U_\epsilon$.

Since U_ϵ is a conical open of S_p it contains a homotopically unique exit-path $\gamma: x_p \rightsquigarrow x_q$. By an argument analogous to the construction of the deformation retraction on [ACG11, pp. 146–149], we obtain a deformation retract $r: X|_{U_\epsilon} \rightarrow X_{x_p}$. Composing with the inclusion $X_{x_q} \hookrightarrow X|_{U_\epsilon}$, we obtain a projection map $r_{p,q}: X_{x_q} \rightarrow X_{x_p}$ with as kernel the vanishing cycle of X_{x_p} . This gives us a diagram:

$$\begin{array}{ccc}
 (\mathbf{R}^1 \pi_* \mathbb{Q})_{x_p} & \xrightarrow{\rho_{\mathcal{F}}(\gamma)} & (\mathbf{R}^1 \pi_* \mathbb{Q})_{x_q} \\
 \uparrow & & \uparrow \\
 H^1(\pi^{-1}(U); \mathbb{Q}) & \xrightarrow{(i_{\pi^{-1}(V)}^*)} & H^1(\pi^{-1}(V); \mathbb{Q}) \\
 \downarrow & & \downarrow \\
 H^1(X_{x_p}; \mathbb{Q}) & \xrightarrow{(r_{p,q})^*} & H^1(X_{x_q}; \mathbb{Q}) \\
 \downarrow r^* & \nearrow (i_{x_q})^* & \\
 H^1(X; \mathbb{Q}) & &
 \end{array}$$

The middle square and bottom triangle commute by definition. So it remains to show the top square commutes to prove our desired result about the cospecialization. Since the cocone of germs on an underlying presheaf F of a sheaf $F^\sharp = \mathcal{F}$ is colimiting, it follows by the argument of lemma 1.3.13 that the square commutes. This completes the first part of the proof.

Now consider our working example π . In [ACG11, pp. 146-149], a deformation retract $X_0 \xrightarrow{i} X$ is explicitly constructed and $X_{x_0} \xrightarrow{i_{x_0}} X \xrightarrow{r} X_0$ is the nodal degeneration map $r_{x_0}: X_{x_0} \rightarrow X_{x_0}/c \cong X_0$. Then by the above argument, $\rho_{\mathcal{F}}(\gamma) = i_{x_0}^* \circ r^* = r_{x_0}^*$ which has co-vanishing cycle c hence is the map $1 \mapsto (0, 1)$ in our chosen cohomology basis. $\square$

Recall that the exit-path category is given by

$$\begin{array}{ccc} \mathbb{Z} & & \text{id}_0 \\ \curvearrowright & & \curvearrowright \\ x_0 & \xleftarrow{*} & 0. \end{array}$$

The exodromy representation of $\mathbf{R}^1 \pi_* \underline{\mathbb{Q}}$ can then be written as

$$\begin{array}{ccc} & & i \mapsto \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \\ & & \curvearrowright \\ 1 & & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \curvearrowright & \xrightarrow{\quad} & \curvearrowright \\ \mathbb{Q} & & \mathbb{Q}^2. \end{array}$$

Where the vertices represent the functor evaluations on the objects and the edges on morphisms (and where we write a single value in the two cases we have a unique exit-path) and where composition is given by matrix multiplication. Computing more complicated exodromy representations and determining effective ways to compute cospecialization maps in a globally compatible manner is the subject of future work. For now, the butterfly flutters on, to a meadow filled with sheaves on stacks.

Chapter 2

Topological Stacks and Their Categories of Sheaves

‘It can follow you like a dog
It brought me to my knees
They got a skin and they put me in
They got a skin and they put me in’
🎵 *Bodysnatchers* — Radiohead

In this chapter, we shift our focus to topological stacks and their associated categories of sheaves. Ultimately, we also extend the exodromy correspondence to this generalized setting. In the first section, we motivate stack theory through the lens of moduli problems. When given a moduli problem, one initially asks if it can be faithfully represented. If this is not the case, we make the problem its own solution; by treating the moduli problem as a geometric object in its own right, we arrive at the definition of a stack. Even though the problem is its own solution, the saying “every solution creates a new problem” still applies: we are forced to work with the category of stacks, which is a 2-category, adding a layer of subtlety to any construction involving stacks. After covering the basics of the theory of topological stacks, we define sheaves on stacks. Every topological stack $\mathfrak{X}$ comes equipped with a “versal family” x , which is a morphism $\underline{S} \xrightarrow{x} \mathfrak{X}$ with some special properties. We can associate a groupoid $\Gamma \rightrightarrows S$ to this versal family. We will see that, via a result implicit in the descent theory of stacks, the category of sheaves on $\mathfrak{X}$ can be expressed as the 2-equalizer of the symmetry groupoid under the pullback functor $(-)^*$:

$$\mathbf{Sh}(\mathfrak{X}) \rightarrow \mathbf{Sh}(\Gamma) \rightrightarrows \mathbf{Sh}(S).$$

We conclude the chapter by deriving an exodromy correspondence and a proper base change theorem, both of which follow naturally from this descent result. This chapter is a recap of the theory and contains no new material, except for the making explicit of the exodromy correspondence on stacks.

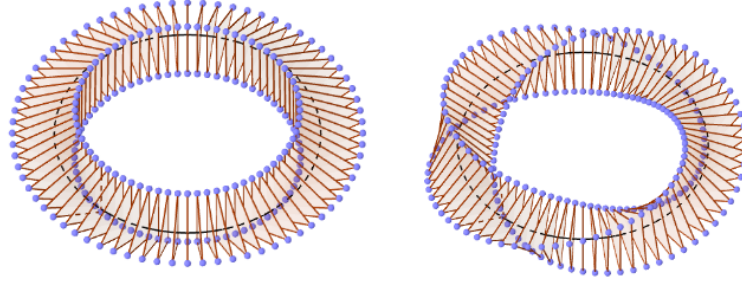


Figure 2.1: Moduli of triangles, family and its twisted version. [Gvi15, Figure 2.4]

2.1 A Motivation for Stack Theory

In this section we motivate stack theory through the lens of moduli problems.¹ We start by defining what a moduli problem is. We are given some collection of objects (e.g. stable nodal curves), some notion of families of such objects over some base category $\mathcal{C}$ (e.g. flat proper families over $\mathbf{Sch}_{\mathbb{C}}$) and a way to pull such families back along morphisms in $\mathcal{C}$. These data are packaged as a functor

$$F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}.$$

Which assigns to every base S in $\mathcal{C}^{\text{op}}$ the set

$$\{\text{families of objects over } S\} / \cong$$

of isomorphism classes of families over S . On morphisms $f: S' \rightarrow S$ maps a family $[X]$ over S to a family $[f^*X]$ over S' . We call F the **moduli functor** or **moduli problem**. We desire to construct an object that classifies all such families.

Definition 2.1.1 (Universal Family). Let $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ be a moduli functor. A family $U \rightarrow M$ is called a **universal family** (and M a **fine moduli space**) if for any base S and any family $X \rightarrow S$, there exists a unique morphism $f: S \rightarrow M$ such that $X \cong f^*U$. $\diamond$

Such a universal family $U \rightarrow M$ exists if and only if the moduli functor F is representable by M [Gvi15, Section 2.3]. It can often be the case that moduli problems are not representable. This is often attributed to there being too many automorphisms. Consider the moduli problem whose objects are triangles and whose families are fiber bundles of triangles over $\mathbf{Top}$ (for more details see [Gvi15, Section 2.2]). A $\frac{2\pi}{3}$ rotation is a non-trivial automorphism of an equilateral triangle. By twisting a trivial family of equilateral triangles using this automorphism, we produce two families which are locally identical but differ globally (see fig. 2.1).

In particular, this contradicts the existence of a universal family, hence this moduli problem has no fine solution. Moduli stacks aim to provide a solution to such moduli problems by directly adding the automorphisms back in. We begin by introducing the moduli problem for stable curves $\overline{\mathcal{M}}_{g,n}$. To begin with, we need to define the objects/families in the moduli problem.

Definition 2.1.2. A **curve** is a proper integral connected one-dimensional scheme over an algebraically closed field k . An **n -marked curve** is a pair $(C; p_i)$ consisting of a curve C and markings $p_i: \text{Spec } k \rightarrow C$.

¹See also [Beh12, Chapter 1] for a good motivation for stacks through the moduli of triangles.

An n -marked curve $(C; p_i)$ is **stable** if $\text{Aut}(C; p_i)$, the group of automorphisms on C fixing the markings p_i , is finite. A curve is **smooth** if it is regular everywhere and **nodal** if it is regular everywhere except at finitely many points P_i at which the completions of the local rings $\widehat{\mathcal{O}}_{C, P_i}$ are isomorphic to $k[[x, y]]/(xy)$ as k -algebras.

An n -**marked stable nodal curve of genus g** over a scheme S is a pair $(\xi: \chi \rightarrow S, (p_i: S \rightarrow \chi))$ consisting of a flat proper morphism ξ and n sections $p_i: S \rightarrow \chi$ such that the geometric fibers χ_s are n -marked stable nodal curves of genus g with markings $p_i(s)$ which lie in the smooth locus and are pairwise distinct. $\diamond$

In algebraic geometry, one class of schemes of interest are the smooth curves. Each smooth curve can be assigned a genus. One can hope to classify all smooth curves according to their genus. To this end, we construct an object that contains the data of all these curves. Instead of considering only the individual curves over $\mathbb{C}$, we formulate the problem functorially by associating to each scheme S over $\mathbb{C}$ the flat families of curves over S . Initially, we attempt to prove the representability of the associated moduli functor $\bar{F}_{g,n}: \mathbf{Sch}_{/\mathbb{C}}^{\text{op}} \rightarrow \mathbf{Set}$ which associates to each $\mathbb{C}$ -scheme S the set of isomorphism classes of families of stable n -pointed² genus g curves over S ,

$$\{\chi \rightarrow S \mid \chi \text{ is a stable } n\text{-pointed genus } g \text{ nodal curves over } S\} / \cong.$$

This functor is not always representable, an argument is given in [ACG11, pp. 266-267]. As such, we are motivated to construct a new object $\overline{\mathcal{M}}_{g,n}$ which does solve the moduli problem. This can be achieved using the theory of stacks.

2.1.1 Moduli stacks

As mentioned before, the stack approach entails enriching the moduli problem by adding back in the automorphisms. Explicitly, for a set valued moduli functor $F: \mathbf{Sch}_{/\mathbb{C}}^{\text{op}} \rightarrow \mathbf{Set}$ and a scheme S over $\mathbb{C}$, this entails replacing the set of isomorphism classes $F(S)$ with the groupoid of families over S and their isomorphisms. We define the pullback of such families using the fibered product, and hence, on morphisms, $F(f)$ is given by the pullback of families along f . Since fibered products of such families are defined up to isomorphism the preservation of composition of morphisms through F is no longer strict. Strictly speaking F is what is called a pseudofunctor (see [Sta26, Definition 003N]). To avoid working with pseudofunctors we instead consider stacks as categories with certain properties (namely, fibered categories in groupoids with some descent properties; this is made explicit in the next section). We make this transition precise below.

Construction 2.1.3 (Grothendieck's Construction). Given a $(2, 1)$ -pseudofunctor $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Grpd}$, we obtain a category $\underline{F}$ defined by:

Objects: (U, x) such that $U \in \text{Ob}(\mathcal{C})$ and $x \in F(U)$.

Morphisms: A morphism $(U, x) \rightarrow (V, y)$ is a pair (f, ϕ) of morphisms $f: U \rightarrow V$ in $\mathcal{C}$ and $\phi: x \rightarrow F(f)(y)$ in $F(U)$.

Composition: Define composition as $(g, \psi) \circ (f, \phi) := (g \circ f, F(f)(\psi) \circ \phi)$.

²We work with n -marked curves in order to ensure stability in cases where it would otherwise fail, thereby enlarging the class of curves that can be considered stable. For example, in genus 1, adding a marked point ensures stability.

If $F = y_S$ then it is denoted $\underline{S}$. ◇

Example 2.1.4. Given a scheme S , its functor of points y_S (using the notation of the Yoneda embedding) is a sheaf. Applying Grothendieck's construction to this functor, one obtains the associated category, $\underline{S}$ where $\text{Ob}(\underline{S}(U)) = \{f: U \rightarrow S : f \text{ is a morphism of schemes}\}$. The morphisms in $\underline{S}$ are just the morphisms $U \rightarrow V$ over S and composition is given by the ordinary composition of morphisms of schemes. ◇

So to solve the moduli problem we want a category $\mathcal{M}$ such that for any scheme S over $\mathbb{C}$ we have an isomorphism of groupoids

$$\text{Hom}(\underline{S}, \mathcal{M}) \cong \{\text{families of genus } g \text{ } n\text{-marked stable nodal curves over } S\}$$

where on the right-hand side we mean the groupoid whose objects are the families and whose morphisms are the isomorphisms of such families over S . So define the groupoid valued functor,

$$\tilde{F}_{g,n}: S \mapsto \{\text{families of genus } g \text{ } n\text{-marked stable nodal curves over } S\}$$

where the category on the right-hand side has as morphisms the S -isomorphisms of stable nodal curves preserving the marked sections. By the 2-Yoneda lemma for the class of such categories (see lemma 2.2.14; the categories being fibered categories, which we define in the next section) and the Grothendieck construction, we have

$$\text{Hom}(\underline{S}, \tilde{F}_{g,n}) \cong \tilde{F}_{g,n}(S) \cong \tilde{F}_{g,n}(S). \quad (2.1)$$

Showing that the fibered category $\tilde{F}_{g,n}$ represents the moduli problem.

Definition 2.1.5. We call the stack $\overline{\mathcal{M}}_{g,n} := \tilde{F}_{g,n}$ the **moduli stack of n -marked genus g stable nodal curves**. ◇

This is equivalent to the definition on [ACG11, p. 281]. The functor $\tilde{F}_{g,n}$ is a 2-sheaf which means that we may call $\tilde{F}_{g,n}$ a **moduli stack**. In fact, it is not only a moduli stack but a moduli stack whose diagonal $\tilde{F}_{g,n} \rightarrow \tilde{F}_{g,n} \times \tilde{F}_{g,n}$ is representable (see [Gvi15, Definition 5.1]) which enables us to naturally extend synonyms of schemes which are stable under base change to $\overline{\mathcal{M}}_{g,n}$. In summary, we have the following two maps ([ACG11, p. 285])

$$\overline{\mathcal{M}}_{g,n} \xrightarrow{\alpha} \overline{F}_{g,n} \xrightarrow{\beta} M_{g,n}.$$

We define these morphisms explicitly. Let $S \in \mathbf{Sch}_{/\mathbb{C}}$ and consider any n -marked genus g stable nodal curve $\xi: \chi \rightarrow S$ over S , that is any object of $\overline{\mathcal{M}}_{g,n}(S)$. The map α is given by $\alpha(S)(\xi) = [\xi]$ where $[\xi]$ is the isomorphism class of ξ and for any morphism $f: \xi \rightarrow \xi'$, $\alpha(S)(f) = \text{id}_{[\xi]}$. Given a class $[\xi: \chi \rightarrow S]$ we define $\beta(S)([\xi])$ as the morphism $\underline{S} \rightarrow \overline{M}_{g,n}, s \mapsto [\chi_s]$ obtained by applying the universal property of the moduli stack (or the stacky Yoneda lemma: $[\xi]$ is a family of S but by stacky Yoneda these correspond to morphisms $\underline{S} \rightarrow \overline{\mathcal{M}}_{g,n}$).

In general, α fails to be an isomorphism since $\overline{F}_{g,n}$ does not keep track of automorphisms and therefore $\overline{F}_{g,n}$ fails to be a stack. Moreover, β fails to be an isomorphism, in general, as this is equivalent to an

isomorphism of functors $\bar{F}_{g,n} \xrightarrow{\sim} y_{\bar{M}_{g,n}}$. Yet $\bar{M}_{g,n}$ does not generally represent $\bar{F}_{g,n}$. Both obstructions boil down to the non-existence of the universal family over $\bar{M}_{g,n}$, so we can say that the non-existence of the universal family necessitates the passage to the category of stacks.

2.2 The Theory of Topological Stacks

In this section we define topological stacks and the theory necessary to work with them. We closely follow the notes by Behrend [Beh12].

2.2.1 Topological groupoids

We start by defining topological groupoids.

Definition 2.2.1 (Groupoid). A **groupoid** is a small category $\mathcal{C}$ in which every morphism is an isomorphism. Equivalently, a groupoid Γ consists of a set of objects Γ_0 and a set of morphisms Γ_1 , and the set of composable morphisms $\Gamma_2 = \{(g, f) \in \Gamma_1^2 \mid s(g) = t(f)\}$ equipped with structure maps:

- **source and target:** $s, t: \Gamma_1 \rightarrow \Gamma_0$,
- **identities:** $e: \Gamma_0 \rightarrow \Gamma_1$ such that $s \circ e = t \circ e = \text{id}_{\Gamma_0}$,
- **groupoid operation:** $\mu: \Gamma_2 \rightarrow \Gamma_1$, denoted $(g, f) \mapsto g * f$,
- **inversion:** $i: \Gamma_1 \rightarrow \Gamma_1$, denoted $f \mapsto f^{-1}$,

satisfying the usual associativity, identity ($f * e_{s(f)} = f = e_{t(f)} * f$), and inverse ($f^{-1} * f = e_{s(f)}$ and $f * f^{-1} = e_{t(f)}$) axioms. $\diamond$

Definition 2.2.2. A **topological groupoid** is a groupoid Γ in which Γ_0 and Γ_1 are topological spaces and the maps s, t, e, μ, ϕ are continuous. $\diamond$

Example 2.2.3. Let X be a topological space and $G \curvearrowright X$ be a group action that is continuous when we endow G with the discrete topology. The transformation groupoid $G^{\text{disc}} \times X \xrightarrow[\sigma]{\text{pr}_2} X$ is a topological groupoid. $\diamond$

2.2.2 Principal G -bundles

In this section we discuss G -bundles which will allow use to define quotient stacks needed in our calculations.

Definition 2.2.4 (Group objects, actions, and equivariance). Let $\mathcal{C}$ be a category with finite products.

- A **group object** in $\mathcal{C}$ is an object G in $\mathcal{C}$ whose functor of points $\text{Hom}_{\mathcal{C}}(-, G)$ factorizes through the category of groups **Grp**.
- An **action** of a group object G on an object X in $\mathcal{C}$ is a morphism $X \times G \rightarrow X$ such that, for every T in $\mathcal{C}$ $X(T) \times G(T) \rightarrow X(T)$ defines a right $G(T)$ -action on $X(T)$.

- A morphism $f: X \rightarrow X'$ between objects equipped with G -actions is **G -equivariant** if for every T in $\mathcal{C}$ the morphism $f(T): X(T) \rightarrow X'(T)$ is $G(T)$ -equivariant. $\diamond$

Definition 2.2.5 (Principal G -bundles). Let $(\mathcal{C}, J)$ be a site with fiber products and a terminal object, and let G be a group object in $\mathcal{C}$. Let S be an object in $\mathcal{C}$.

- A **G -bundle** is a morphism $p: E \rightarrow S$ equipped with an action $E \times G \rightarrow E$ preserving the fibers.
- Given a morphism $f: S' \rightarrow S$ and a G -bundle $E \rightarrow S$ the **pullback along f** , written f^*E , is $E \times_S S'$ and inherits the G -bundle structure from the action of G on E .
- A G -bundle E is **principal** if there exists a covering family $\{f_i: U_i \rightarrow S\}$ in J such that f_i^*E is isomorphic to the trivial G -bundle $U_i \times G \rightarrow U_i$, i.e. E is locally trivial. $\diamond$

Remark 2.2.6. If $E \rightarrow S$ is a principal G -bundle then so is $f^*E \rightarrow S'$ [Gvi15, Example 3.13]. $\diamond$

2.2.3 Categories Fibered in Groupoids

To begin with let $\mathcal{C}$ be a category.

Definition 2.2.7 (Fibred category). Let $p: \mathfrak{X} \rightarrow \mathcal{C}$ be a category over $\mathcal{C}$.

- Consider a morphism $\phi: x' \rightarrow x$ lying over a morphism $f: S' \rightarrow S$.

$$\begin{array}{ccc} x & \xrightarrow{\phi} & x \\ p \downarrow & & \downarrow p \\ S & \xrightarrow{f} & S \end{array}$$

Then ϕ is **cartesian** if for any object z in $\mathfrak{X}$ and every morphism $\alpha: p(z) \rightarrow S'$ together with a morphism $\beta: z \rightarrow x$ such that $p(\beta) = f \circ \alpha$, there exists a unique morphism $\gamma: z \rightarrow x'$ such that $\phi \circ \gamma = \beta$ and $p(\gamma) = \alpha$. That is, making the following diagram commute.

$$\begin{array}{ccccc} z & \xrightarrow{\gamma} & x' & \xrightarrow{\phi} & x \\ p \downarrow & & p \downarrow & & \downarrow p \\ p(z) & \xrightarrow{\alpha} & S' & \xrightarrow{f} & S \end{array}$$

Whenever ϕ is cartesian, x' is a **pullback of x along f** denoted f^*x .

- $\mathfrak{X}$ is a fibered category if for any morphism $f: S' \rightarrow S$ in $\mathcal{C}$ there exists a cartesian square

$$\begin{array}{ccc} x' & \xrightarrow{\phi} & x \\ p \downarrow & & \downarrow p \\ S' & \xrightarrow{f} & S. \end{array}$$

A **morphism** of fibered categories is a functor over $\mathcal{C}$ preserving cartesian morphisms. $\diamond$

Proposition 2.2.8 (The pullback functor). [Beh12, Exercise 2.4] Let $\mathfrak{X}$ be a fibered category over $\mathcal{C}$. Then any map $f: S' \rightarrow S$ in $\mathcal{C}$ induces a functor $f^*: \mathfrak{X}_S \rightarrow \mathfrak{X}_{S'}$ unique up to natural isomorphism. Moreover $(gf)^* = f^*g^*$. $\diamond$

Definition 2.2.9. A **category fibered in groupoids** (abbreviated CFG) over $\mathcal{C}$ is a fibered category $p: \mathfrak{X} \rightarrow \mathcal{C}$ such that for every object S in $\mathcal{C}$ the fiber category $\mathfrak{X}_S$ is a groupoid. Morphisms of CFGs are morphisms of fibered categories over $\mathcal{C}$. $\diamond$

Let $\mathfrak{X}$ be a category fibered in groupoids. Using the axiom of choice for collections or a suitable context based construction, we can choose a pullback in $\mathfrak{X}$ of an object $x \in \mathfrak{X}(T)$ along some $T' \xrightarrow{f} T$ for every T', T and f functorially (recall proposition 2.2.8).

From now on we assume a suitable choice of pullbacks has been made.

Definition 2.2.10 (Terminology). We will call an object $x \in \mathfrak{X}(T)$ a **family of $\mathfrak{X}$ over T** and may denote it by x/T . For a family x/T of $\mathfrak{X}$, a point $t \in T$ corresponds to a map $*$ $\xrightarrow{t}$ T ; the pullback t^*x is denoted x_t and called the **family member at t** . $\diamond$

Example 2.2.11. The functor $\underline{X} = \text{Hom}(-, X)$ defines a category fibered in groupoids as discussed in the previous section. It has X as its fine moduli space and has fiber functor $\underline{X} \rightarrow \mathbf{Top}$ defined by $(T \rightarrow X) \mapsto T$. $\diamond$

Recall we defined both what a CFG is and their morphisms in definition 2.2.9.

Definition 2.2.12 (Isomorphic CFGs). A **2-isomorphism** of morphisms of CFGs $F, G: \mathfrak{X} \rightarrow \mathcal{Y}$ is a natural transformation $\theta: F \Rightarrow G$ such that for all x/T in $\mathfrak{X}$, $\theta(x): F(x) \rightarrow G(x)$ lies over id_T in $\mathcal{Y}$. Two CFGs $\mathfrak{X}$ and $\mathcal{Y}$ are **isomorphic** if there exist morphisms $F: \mathfrak{X} \rightarrow \mathcal{Y}$ and $G: \mathcal{Y} \rightarrow \mathfrak{X}$ and 2-isomorphisms $\theta: G \circ F \Rightarrow \text{id}_{\mathfrak{X}}$ and $\eta: F \circ G \Rightarrow \text{id}_{\mathcal{Y}}$. $\diamond$

In other words, the collection of categories fibered in groupoids defines a $(2, 1)$ -category. It can essentially be thought of as a category of categories with functors and natural isomorphisms.

Lemma 2.2.13. A morphism of CFGs F is an iso if and only if it is an equivalence of categories as a functor. A natural transformation is a 2-isomorphism if and only if it is a natural isomorphism.

Proof. For the first statement, recall that an equivalence of categories can be characterized as an adjunction where the unit and counit maps are isomorphisms. The second statement follows from unravelling the definitions. $\square$

Lemma 2.2.14 (The 2-Yoneda lemma). [Beh12, Exercise 2.9] Let $\mathfrak{X}$ be a CFG. If $\mathfrak{X}$ has chosen pullbacks, then for any topological space T there is a correspondence

$$\text{Hom}(\underline{T}, \mathfrak{X}) \xrightarrow{1-1} \mathfrak{X}(T).$$

Moreover, the embedding, $\mathbf{Top} \hookrightarrow \mathbf{CFGs}$ into the category of CFGs is fully faithful. $\diamond$

Fibred products of CFGs

In this section we define fibred products, an important construction we will use when doing computations in the final chapter of this thesis.

Definition 2.2.15. Let $F: \mathfrak{X} \rightarrow \mathcal{Z}$ and $G: \mathcal{Y} \rightarrow \mathcal{Z}$ be morphisms of CFGs. The **fibred product** $\mathfrak{N}$ is defined as,

$$\mathfrak{N}(T) := \{(x, \phi, y) | x \in \mathfrak{X}(T), y \in \mathcal{Y}(T) \text{ and } \phi: F(x) \rightarrow G(y) \text{ is an isomorphism in } \mathcal{Z}(T)\}.$$

With morphisms $(\alpha, \beta): (x', \phi', y') \rightarrow (x, \phi, y)$ over $f: T' \rightarrow T$ being an α over f in $\mathfrak{X}$ and a β over f in $\mathcal{Y}$ such that $G(\beta) \circ \phi' = \phi \circ F(\alpha)$. $\diamond$

Definition 2.2.16 (2-Cartesian). [Gvi15, Definition 3.28] Let $\mathcal{C}$ be a $(2, 1)$ -category, e.g. the category fibred in groupoids.

1. A square,

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{F} & \mathcal{Y} \\ G \downarrow & & \downarrow F' \\ \mathfrak{X} & \xrightarrow{G'} & \mathcal{Z} \end{array}$$

is **2-commutative** if there exists a natural isomorphism $H: F' \circ F \xRightarrow{\sim} G' \circ G$.

2. A 2-commutative square

$$\begin{array}{ccc} \mathfrak{X} \times_{\mathcal{Z}} \mathcal{Y} & \xrightarrow{\text{pr}_{\mathcal{Y}}} & \mathcal{Y} \\ \text{pr}_{\mathfrak{X}} \downarrow & \nearrow H & \downarrow G \\ \mathfrak{X} & \xrightarrow{F} & \mathcal{Z} \end{array}$$

is **2-cartesian** if: For any $\mathcal{T}$ in $\mathcal{C}$ with maps into $\mathfrak{X}$ and $\mathcal{Y}$ such that

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow G \\ \mathfrak{X} & \xrightarrow{F} & \mathcal{Z} \end{array}$$

is 2-commutative, there exists a morphism $\gamma: \mathcal{T} \rightarrow \mathfrak{X} \times_{\mathcal{Z}} \mathcal{Y}$ such that

$$\begin{array}{c} \mathcal{T} \xrightarrow{\quad} \mathfrak{X} \times_{\mathcal{Z}} \mathcal{Y} \xrightarrow{\text{pr}_{\mathcal{Y}}} \mathcal{Y} \\ \searrow \quad \downarrow \text{pr}_{\mathfrak{X}} \quad \nearrow H \quad \downarrow G \\ \mathfrak{X} \xrightarrow{F} \mathcal{Z} \end{array}$$

2-commutes and such that for any other $\gamma': \mathcal{T} \rightarrow \mathfrak{X} \times_{\mathcal{Z}} \mathcal{Y}$ there exists a unique 2-isomorphism $\gamma \Rightarrow \gamma'$ such that the whole diagram 2-commutes. $\diamond$

Lemma 2.2.17. [Beh12, p. 77] The 2-commutative diagram of CFGs,

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\text{pr}_{\mathcal{Y}}} & \mathcal{Y} \\ \text{pr}_{\mathfrak{X}} \downarrow & \nearrow H & \downarrow G \\ \mathfrak{X} & \xrightarrow{F} & \mathcal{Z} \end{array}$$

where $H(x, \phi, y) = \phi$ is 2-cartesian. So we may write $\mathfrak{X} \times_{\mathcal{Z}} \mathcal{Y}$ for $\mathfrak{M}$. ◇

We now define the symmetry groupoid, an important construction which turns out to be our first example of a 2-fibered product. We make an abrupt shift from the functor of points viewpoint to a pointwise one to honor the conventions of [Beh12].

Definition 2.2.18 (The symmetry groupoid). [Beh12, Example 2.10] Let $\mathfrak{X}$ be a category fibered in groupoids (CFG), and let $x \in \mathfrak{X}(T)$. The **symmetry groupoid of x** is the groupoid Γ whose objects are the family members x_t of x and whose morphisms are triples (t, t', ϕ) , where $\phi: x_t \rightarrow x_{t'}$ is a morphism in $\mathfrak{X}(*)$. The **canonical topology** on Γ is the finest topology such that for any space S , maps $f, g: S \rightarrow T$, and $\phi: f^*x \rightarrow g^*x$, the map

$$S \rightarrow \Gamma_1, \quad s \mapsto (f(s), g(s), s^*\phi)$$

is continuous, where $s^*\phi: x_{f(s)} \rightarrow x_{g(s)}$. ◇

For the functor of points version of the next definition see [Sta26, Tag 02ZB].

Definition 2.2.19. Let $\mathfrak{X}$ be a category fibered in groupoids, let T, S topological spaces, and let $x/T, y/S$ be families. The **space of isomorphisms** is the set

$$\underline{\text{Isom}}(x, y) = \{(t, s, \phi) \mid t \in T, s \in S, \phi: x_t \rightarrow y_s \text{ is an isomorphism in } \mathfrak{X}(*)\},$$

endowed with the finest topology such that for every span $T \leftarrow U \rightarrow S$ (which is just a pair of maps $U \rightarrow T$ and $U \rightarrow S$ in **Top**) and every isomorphism $\phi: f^*x \rightarrow g^*y$ in $\mathfrak{X}(U)$, the map $U \rightarrow \underline{\text{Isom}}(x, y)$ given by $u \mapsto (f(u), g(u), \phi_u)$ is continuous. ◇

Remark 2.2.20. So essentially the opens of $\underline{\text{Isom}}(x, y)$ are collections of isomorphisms/symmetries of members which glue to some isomorphism of families on indexing space U . ◇

Definition 2.2.21. A **pre-stack** is a CFG $\mathfrak{X} \xrightarrow{F} \mathbf{Top}$ such that for all families x/S and y/T

(Isomorphisms of members glue over the source/target) The source $s: \underline{\text{Isom}}(x, y) \rightarrow S$ and target $t: \underline{\text{Isom}}(x, y) \rightarrow T$ maps are continuous.

(Isomorphisms of families are characterized by continuous maps into $\underline{\text{Isom}}(x, y)$) For any continuous map $\alpha: U \rightarrow \underline{\text{Isom}}(x, y)$, there exists a unique isomorphism of families $\phi: x|_U \rightarrow y|_U$ such that $\phi_u = \alpha(u)$. ◇

This is a slightly stronger notion of pre-stack used in [Beh12]. The pre-stack conditions govern how members glue into families. In fact the pre-stack conditions can be expressed as $\underline{\text{Isom}}(x, y)$ being a certain fibered product. A morphism of pre-stacks is a morphism of CFGs between two pre-stacks.

Proposition 2.2.22. [Beh12, Exercise 2.13] Let $\mathfrak{X}$ be a CFG. Then $\mathfrak{X}$ is a pre-stack if and only if for any two families x/T and y/S the diagram,

$$\begin{array}{ccc} \underline{\text{Isom}}(x, y) & \xrightarrow{s} & \underline{S} \\ t \downarrow & \nearrow \phi & \downarrow y \\ \underline{T} & \xrightarrow{x} & \mathfrak{X} \end{array}$$

is 2-cartesian.

Remark 2.2.23. Let $\mathfrak{X}$ be a pre-stack and $x: \underline{U} \rightarrow \mathfrak{X}$ a family, then $\underline{U} \times_{\mathfrak{X}} \underline{U}$ is the symmetry groupoid Γ of x since $\underline{\text{Isom}}(x, x)$ are the arrows of Γ when the pre-stack conditions hold. $\diamond$

A stack is a pre-stack with a certain 2-sheaf condition, this condition is made explicit with descent data.

Definition 2.2.24 (Descent data). Let $\{x_i/U_i\}_i$ be a collection of families. A collection $\{\phi_{ij}\}$ is **descent data** if $\phi_{ij}: x_i|_{U_{ij}} \rightarrow x_j|_{U_{ij}}$ are isomorphisms satisfying the cocycle condition. We say that descent data $\{\phi_{ij}\}$ is **effective** if there exists a family $x/\cup U_i$ unique up to unique isomorphism such that $x|_{U_i}/U_i$ is isomorphic to x_i . $\diamond$

Definition 2.2.25. A pre-stack $\mathfrak{X}$ over **Top** is a **stack** if all descent data is effective. $\diamond$

2.2.4 Versal Families and Topological Stacks

In this chapter, we define the notion of a versal family. If our moduli problem does not admit a universal family, which as we saw is the reason for working with the moduli stacks, the next best thing to hope for is a versal family. A versal family of $\mathfrak{X}$ is a family in $\mathfrak{X}$ that locally describes every other family. Explicitly:

Definition 2.2.26. Let $\mathfrak{X}$ be a CFG. A family x/T is **versal** if every family can be **locally pulled back from** x meaning

- (*) for any y/S and $s \in S$ there exists an open neighborhood $s \in U \subset S$ and an $f: U \rightarrow T$ such that $f^*x \cong y|_U$

and if the pre-stack conditions hold for $\underline{\text{Isom}}(x, x)$. $\diamond$

Definition 2.2.27. A **topological stack** is a stack $\mathfrak{X}$ which admits a versal family. We call it an **Deligne–Mumford topological stack** if it admits an étalé versal family (see [Beh12, Definition 2.42], otherwise not important so we do not give the definition). $\diamond$

As discussed in the first section of this chapter, stacks can be used to solve moduli problems. Indeed, every topological stack solves the moduli problem

$$\mathbf{Top}^{\text{op}} \rightarrow \mathbf{Set}, \quad T \mapsto (\mathfrak{X}(T))^{\text{Set}}$$

which on morphisms is given by the pullback construction. Another way to solve the moduli problem is using the coarse moduli.

Definition 2.2.28 (Coarse Moduli, [Beh12, Definition 1.18]). Let $F: \mathbf{Top}^{\text{op}} \rightarrow \mathbf{Set}$ be a moduli problem. A topological space M is a **coarse moduli space** for F if:

1. There is a correspondence between the points of M and the isomorphism classes of objects in question $F(*)$.
2. For every topological space T and every family y in $F(T)$, the map

$$T \rightarrow M, \quad t \mapsto y|_t$$

is continuous.

3. Given any other topological space N for which the previous conditions hold, there exists a unique continuous map $M \rightarrow N$ compatible with the bijection of sets $F(*) \cong N$. $\diamond$

The next proposition tells us how to obtain the coarse moduli for this problem.

Proposition 2.2.29 (The Coarse Moduli of a Topological Stack). Let $\mathfrak{X}$ be a topological stack with versal family x/T and associated symmetry groupoid Γ . Then $\Gamma \xrightarrow{(s,t)} T \times T$ defines an equivalence relation on T . The topological quotient $T / \sim$ is a coarse moduli space of $\mathfrak{X}$.

Proof. This is [Beh12, Exercise 2.21]. The relation is reflexive since every member of the versal family x/T is isomorphic to itself via id_{x_t} . Since isomorphisms invert and compose, the relation is also symmetric and transitive. So $T / \sim$ is a topological space.

The equivalence relation quotients out redundant members of the versal family. So since versal families locally describe every family they every member of every possible family, so elements of $T / \sim$ are in one-to-one correspondence with isomorphism classes of the members of the moduli problem.

For the second property of coarse moduli spaces to hold, we want to show $T / \sim$ has the finest topology such that all associated moduli maps $\mu_y : S \rightarrow T / \sim$ are continuous. This follows from,

Claim 2.2.29.1. $T / \sim$ is endowed with the finest topology such that $\mu_y : S \rightarrow T / \sim$ is continuous for all y/S if and only if $T / \sim$ has the quotient topology.

Proof of Claim. By the above, we know that μ_x must locally factor through $T \rightarrow T / \sim$. So the topology on $T / \sim$ must be coarser than the quotient topology. From the first sentence it also follows that for the versal family x/T the associated moduli map μ_x is the quotient map $T \rightarrow T / \sim$. By the local construction above and the fact that the $U_i \rightarrow T$ is continuous since x/T is a versal family, it follows that all μ_y are continuous. So this topology in which all μ_y are continuous is the finest topology such that the quotient map $T \rightarrow T / \sim$ is continuous, i.e. the quotient topology. $\square_{\text{Claim}}$

So indeed, $T / \sim$ is a coarse moduli space of $\mathfrak{X}$. $\square$

Example 2.2.30. The moduli stack $\overline{\mathcal{M}}_{g,n}$ has open interior $\mathcal{M}_{g,n}$ which solves the moduli problem

$$S \mapsto \{X \xrightarrow{f} S : f \text{ is a smooth family of } n\text{-marked genus } g \text{ curves}\}.$$

Consider the family $\mathcal{E}/\mathbb{H}$ of $\mathcal{M}_{1,1}$ where for any $\tau \in \mathbb{H}$ the member $\mathcal{E}_\tau$ is the elliptic curve corresponding to lattice $\mathbb{Z} \oplus \tau\mathbb{Z}$ marked by τ . This is a versal family for the stack $\mathcal{M}_{1,1}$. By uniformization [Sil09] these elliptic curves up to isomorphism form the analytic space $M_{1,1}$ which by the previous proposition is the coarse moduli space of $\mathcal{M}_{1,1}$. $\diamond$

Finally, we define what a quotient stack is, which will be of importance to us later.

Definition 2.2.31 (Quotient Stack). Let X be a topological space and G a topological group acting on X . The **quotient stack** is the fibered category $[X/G]$ whose families over T are diagrams,

$$\begin{array}{ccc} P & \xrightarrow{f} & X \\ \pi \downarrow & & \\ T & & \end{array}$$

where π is a principal G -bundle and f is a G -equivariant map. $\diamond$

See [Beh12, p. 86] for the argument that $[X/G]$ is a stack.

2.3 Sheaves on Stacks

In this section we define sheaves on stacks, prove some descent results, and the proper base change theorem for stacks. First off we give two equivalent definitions of a sheaf on a stack. We follow the definitions in [Jan24, Section 2].

Definition 2.3.1. Let $\mathfrak{X}$ be a stack with representable diagonal. Consider the full subcategory $\mathcal{LH}(\mathfrak{X})$ of $\mathbf{Stk}_{/\mathfrak{X}}$ whose objects are local homeomorphisms $\underline{U} \rightarrow \mathfrak{X}$ (which are representable since the diagonal is representable), and whose morphisms are commutative triangles over $\mathfrak{X}$. We endow $\mathcal{LH}(\mathfrak{X})$ with the following Grothendieck topology: a collection $\{U_i \rightarrow U\}_{i \in I}$ in $\mathcal{LH}(\mathfrak{X})$ is a covering family if the induced map of topological spaces $\coprod_i U_i \rightarrow U$ is an effective epimorphism (i.e., a quotient map). For a 1-category $\mathcal{D}$ the $\mathcal{D}$ -valued étalé sheaves on $\mathfrak{X}$ are defined to be the functors $\mathcal{LH}(\mathfrak{X})^{\text{op}} \rightarrow \mathcal{D}$ that satisfy the sheaf condition with respect to the topology. We denote this category by

$$\mathbf{Sh}_{\text{ét}}(\mathfrak{X}) := \mathbf{Sh}(\mathcal{LH}(\mathfrak{X}); \mathcal{D}).$$

$\diamond$

Definition 2.3.2. Let $F: \mathcal{C} \rightarrow \mathbf{Top}$ be a category fibered in groupoids. A $\mathcal{D}$ -valued sheaf on $\mathcal{C}$ is a presheaf $\mathcal{F}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ such that for all $X \in \text{Ob}(\mathbf{Top})$ and $x \in \text{Ob}(F^{-1})(X)$ the presheaf $\mathcal{F}_{X,x}$ on X given by

$$(U \xrightarrow{i} X) \mapsto \mathcal{F}(i^*x)$$

is a $\mathcal{D}$ -valued sheaf. $\diamond$

When the stack admits an étalé versal family (see [Beh12, Definition 2.42]), the two definitions are equivalent.

Lemma 2.3.3. [Jan24, Proposition 2.13] Let $\mathfrak{X}$ be a Deligne–Mumford topological stack. Then there is an equivalence $\Psi_{\mathfrak{X}}: \mathbf{Sh}(\mathfrak{X}) \rightarrow \mathbf{Sh}_{\text{ét}}(\mathfrak{X})$ induced by the inclusion of categories $\text{Open}(T) \hookrightarrow \mathcal{LH}(T)$ for all T in $\mathbf{Top}$.

Remark 2.3.4. Note that the étalé sheaves have all the familiar constructions that sheaves of sites have; this equivalence ensures these constructions also exist for sheaves in the sense of definition 2.3.2. $\diamond$

2.3.1 Constructible Sheaves and the Exit Path Category

We can now define the category of constructible sheaves on a Deligne–Mumford topological stack; we choose to restrict to the class of Deligne–Mumford topological stacks so that we can use both definitions of a sheaf on a stack given in the previous section. Consider a poset P as a topological stack by endowing it with the Alexandrov topology and considering the representable stack $\underline{P}$. This section closely follows a paper by Jansen [Jan24, Sections 3.1, 3.2]

Definition 2.3.5. A **stratification** of $\mathfrak{X}$ over P is a morphism of stacks $s: \mathfrak{X} \rightarrow \underline{P}$. We call $(\mathfrak{X}, P)$ a **stratified stack**. The **strata** of $(\mathfrak{X}, P)$ are the sub-stacks $\mathfrak{X}_p := \mathfrak{X} \times_P \{p\}$. A morphism of stratified stacks $(\mathfrak{X}, P) \rightarrow (\mathcal{Y}, Q)$ consists of a morphism $f: \mathfrak{X} \rightarrow \mathcal{Y}$ and a poset map $\sigma: P \rightarrow Q$ which together respect the stratifications, that is, the following diagram commutes

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{f} & \mathcal{Y} \\ \downarrow & & \downarrow \\ P & \xrightarrow{\sigma} & Q. \end{array} \quad \diamond$$

We similarly define the notion of conically stratified stack.

Definition 2.3.6 (Conically Stratified Topological Stacks). A stratified topological stack $(\mathfrak{X}, P)$ is **conically stratified** if there exists a versal family $\underline{U} \rightarrow \mathfrak{X}$ such that the both spaces in the topological groupoid $\Gamma \rightrightarrows \underline{U}$ are conically stratified under the induced stratifications $\underline{\Gamma} \rightrightarrows \underline{U} \rightarrow \mathfrak{X} \rightarrow \underline{P}$. $\diamond$

Now we can define constructible sheaves on a stratified DM stack.

Definition 2.3.7. Let $\mathcal{C}$ be a 1-category and $\mathfrak{X}$ a topological stack with projection $p: \mathfrak{X} \rightarrow *$. Let $\mathcal{F}$ be a sheaf on $\mathfrak{X}$.

- The sheaf $\mathcal{F}$ is **constant** if it lies in the essential image of $p^*: \mathbf{Sh}(*; \mathcal{C}) \rightarrow \mathbf{Sh}(\mathfrak{X}; \mathcal{C})$.
- The sheaf $\mathcal{F}$ is **locally constant** if there exists a covering $\{h_i: U_i \rightarrow \mathfrak{X}\}$ in $\mathcal{LH}(\mathfrak{X})$ such that each pullback $h_i^* \mathcal{F}$ is constant on U_i .
- If $(\mathfrak{X}, P)$ is a stratified Deligne–Mumford topological stack, then $\mathcal{F}$ is **P -constructible** if for each stratum inclusion $i_p: \mathfrak{X}_p \hookrightarrow \mathfrak{X}$, the sheaf $i_p^* \mathcal{F}$ is locally constant on $\mathfrak{X}_p$. $\diamond$

We can now readily define the category of constructible sheaves on a stratified DM stack.

Definition 2.3.8. The category $\mathbf{Sh}_P^{\text{cons}}(\mathfrak{X}; \mathcal{C})$ is the full subcategory of $\mathbf{Sh}(\mathfrak{X}; \mathcal{C})$ of constructible étalé sheaves. $\diamond$

2.3.2 Descent Results for Sheaves on Stacks

Now we discuss some descent results for sheaves on topological stacks. These results are essential for the computations. Jansen uses descent for closed covers to compute the exit-path category of $\overline{\mathcal{M}}_{g,n}$ via its real-oriented blowup. We then proceed to translate descent for closed covers and [Jan23, Remark 4.3] to obtain a descent result for what we call Γ -equivariant sheaves. We later apply this result directly to compute exodromy representations of sheaves, so accordingly end this section by interpreting Γ -equivariant descent under the exodromy correspondence. The next lemma is the 1-categorical version of [Jan24, Corollary 2.20].

Lemma 2.3.9. Consider,

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{i'} & \mathfrak{X} \\ f' \downarrow & \lrcorner & \downarrow f \\ \mathcal{Z} & \xrightarrow{i} & \mathcal{Y} \end{array}$$

where

- (i) f is proper.
- (ii) $\mathcal{Z}$ is a closed substack.
- (iii) f is an isomorphism when pulled back along $\mathcal{Y} \setminus \mathcal{Z} \hookrightarrow \mathcal{Y}$.

Then, the cartesian square is preserved under $\mathbf{Sh}(-)$, so

$$\mathbf{Sh}(\mathcal{Y}) \xrightarrow{\sim} \mathbf{Sh}(\mathfrak{X}) \times_{\mathbf{Sh}(\mathfrak{X}')} \mathbf{Sh}(\mathcal{Z}). \quad \diamond$$

The following corollary is the 1-categorical version of [Jan24, Corollary 2.21].

Corollary 2.3.10 (Descent for closed covers). Let $\mathfrak{X}$ be an étalé topological stack, P be a finite poset (under inclusion) of closed sub-stacks of $\mathfrak{X}$. Suppose that any intersection of elements of P is a union of elements of P , i.e. for any poset $P' \subset P$ there exists a $P'' \subset P$ such that

$$\bigcap_{S \in P'} S = \bigcup_{T \in P''} T.$$

In particular, if $P' = \emptyset$ then $\mathfrak{X} = \bigcup_P S$. Then

$$\mathbf{Sh}(\mathfrak{X}) \xrightarrow{\sim} \varprojlim_{S \in P^{\text{op}}} \mathbf{Sh}(S).$$

Proof. Follows from induction by applying the previous result. □

2.3.3 Descent for Γ -equivariant Sheaves

Now let $\mathfrak{X}$ be a topological stack with versal family x/S and symmetry groupoid $\Gamma \rightrightarrows S$. The 1-categorical version of [Jan23, Remark 4.3] is that $[X/G] \cong \varinjlim_{\mathbf{BG}} \underline{X}$. By descent of sheaves this implies that,

$$\mathbf{Sh}([X/G]) \cong \mathbf{Sh}(\varinjlim_{\mathbf{BG}} \underline{X}) \cong \varprojlim_{\mathbf{BG}} \mathbf{Sh}(\underline{X}) = \varprojlim_{\mathbf{BG}} \mathbf{Sh}(X).$$

An element on the right-hand side is a cone, but equivalently it is a G -equivariant sheaf, which consists of a sheaf $\mathcal{F}$ and an isomorphism $\phi: (\sigma)^* \mathcal{F} \rightarrow (\text{pr}_2)^* \mathcal{F}$ expressed in terms of the maps from the symmetry groupoid of the stack $(\sigma, \text{pr}_2): G \times X \rightrightarrows X$. This result in fact applies to any topological stack as proven below, giving us a characterization of sheaves on a stack that will be essential in our computations later on.

Definition 2.3.11. Let $\Gamma \rightrightarrows S$ be a topological groupoid. The category of Γ -equivariant sheaves on S denoted $\mathbf{Sh}_{\Gamma}(S)$ consists of pairs $(\mathcal{F}, \phi)$ where $\mathcal{F}$ is a sheaf on S and $\phi: s^* \mathcal{F} \rightarrow t^* \mathcal{F}$ is an isomorphism such that

$$\phi_g \circ \phi_f = \phi_{g \circ f} \text{ for every pair of composable arrows } f, g \in \Gamma.$$

A **morphism** $\eta: (\mathcal{F}, \alpha) \rightarrow (\mathcal{G}, \beta)$ is a natural transformation $\eta: \mathcal{F} \Rightarrow \mathcal{G}$ such that,

$$\begin{array}{ccc} s^* \mathcal{F} & \xrightarrow{\alpha} & t^* \mathcal{F} \\ s^* \eta \downarrow & & \downarrow t^* \eta \\ s^* \mathcal{G} & \xrightarrow{\beta} & t^* \mathcal{G} \end{array}$$

commutes. The full subcategory of constructible Γ -equivariant sheaves is denoted by $\mathbf{Sh}_\Gamma^{\text{cons}}(S)$. ◇

For the next theorem, both the formulation and proof sketch were devised by AI, it is very likely already in existing literature. See, for example [Ols16, Paragraph 9.2.2, Proposition 9.2.13] where Olsson defines a morphism of toposes

$$a: \mathcal{X}_{\text{lis-ét}}|_{X_\bullet} \rightarrow X_{\bullet, \text{ét}}$$

And proves an equivalence between $\mathbf{QCoh}(\mathcal{X})$ and the quasicoherent $\mathcal{O}_{X_{\bullet, \text{ét}}}$ -modules. This result should also follow from the fact that any topological stack $\mathcal{M}$ can be expressed as a the 2-coequalizer of its symmetry groupoid and then using the fact $(-)^*$ functor preserves coequalizers.

Theorem 2.3.12. Let $\mathcal{X}$ be a topological stack with pullbacks (so we either assume the axiom of choice for collection or use some context specific construction to define the pullback), x/S be a versal family and $s, t: \Gamma \rightrightarrows S$ be the symmetry groupoid of x/S . Then there is an equivalence of categories between the sheaves on $\mathcal{X}$ and the Γ -equivariant sheaves on S .

Proof. We construct a functor $F: \mathbf{Sh}(\mathcal{X}) \rightarrow \mathbf{Sh}^\Gamma(S)$. So given a sheaf $\mathcal{F}$ we would like to obtain a pair $(\mathcal{F}_{S,x}, \alpha)$ satisfying the cocycle condition. Let $\mathcal{F}_{S,x}$ be $U \mapsto \mathcal{F}(x|_U)$, which is a sheaf by definition 2.3.2. Since Γ is the transformation groupoid, it is isomorphic to $\underline{\text{Isom}}(x, x)$ (see [Beh12, Example 2.10, Exercise 2.13]), so by the second pre-stack condition, the morphism id_Γ induces a morphism $\phi: s^*x \rightarrow t^*x$ given by $\phi_u = u$ for all $u \in \Gamma$ (which is called the tautological morphism). Define $\alpha(U) = \mathcal{F}(\phi|_U)$, by functoriality of $\mathcal{F}$ this defines a morphism of sheaves, $s^*\mathcal{F}_{S,x} \rightarrow t^*\mathcal{F}_{S,x}$. Moreover, since for every h we have that $\alpha_h = \mathcal{F}(\phi_h) = \mathcal{F}(h)$, the cocycle condition holds by functoriality of $\mathcal{F}$. Therefore, $(\mathcal{F}_{S,x}, \alpha)$ is a Γ -equivariant sheaf.

A natural transformation $\eta: \mathcal{F} \Rightarrow \mathcal{G}$ of sheaves of stacks induces a morphism

$\eta': \mathcal{F}_{S,x} \rightarrow \mathcal{G}_{S,x}$ defined by $\eta'(U) := \eta(x|_U)$. We can verify that this indeed defines a morphism $\eta': (\mathcal{F}, \alpha) \rightarrow (\mathcal{G}, \beta)$. Since this construction preserves compositions and the identity, F is a functor.

The functor F is essentially surjective: Let $(\mathcal{F}_{S,x}, \alpha)$ be a Γ -equivariant sheaf. We begin by defining $\mathcal{F}$.

Let y/T be a family in $\mathcal{X}$. Since x/S is versal there exists an open covering $\{U_i\}$ of T along with morphisms $\{f_i: U_i \rightarrow S\}$ such that $\phi_i: y|_{U_i} \rightarrow f_i^*x$ are isomorphisms. For all i, j we have transition maps $\phi_j \circ \phi_i^{-1}$ on $\mathcal{X}(U_{ij})$. By the pre-stack condition this corresponds to a map $\varphi_{ij}: U_{ij} \rightarrow \Gamma$. Pulling α back along the φ_{ij} produces maps $\theta_{ij}: f_i^*\mathcal{F}_{x,S} \rightarrow f_j^*\mathcal{F}_{x,S}$ on $\mathbf{Sh}(U_{ij})$ which satisfy the cocycle condition. Namely for all $x \in U_{ijk}$,

$$\begin{aligned} (\theta_{jk} \circ \theta_{ij})_x &= (\varphi_{jk}^* \alpha \circ \varphi_{ij}^* \alpha)_x \\ &= \alpha_{\varphi_{jk} \circ \varphi_{ij}(x)} \circ \alpha_{\varphi_{ij}(x)} \end{aligned}$$

but $\alpha_{\varphi_{jk}(\varphi_{ij}(x))} = \alpha_{(\phi_k \circ \phi_j^{-1})_{\phi_j \circ \phi_i^{-1}(x)}}$ and $\alpha_{\varphi_{ij}(x)} = \alpha_{((\phi_j \circ \phi_i^{-1})_x)}$. So by the cocycle condition on α this becomes

$$\begin{aligned} \alpha_{(\phi_k \circ \phi_j^{-1} \circ \phi_j \circ \phi_i^{-1})_x} &= \alpha_{(\phi_k \circ \phi_i^{-1})_x} \\ &= ((\phi_{ik})_* \alpha)_x = (\theta_{ik})_x. \end{aligned}$$

So by descent of sheaves, these glue to a sheaf $\mathcal{F}_{y,T}$. We define $\mathcal{F}(y) := \mathcal{F}_{y,T}(T)$.

Claim 2.3.12.1. $\mathcal{F}_{y,T}$ is independent of the choice of trivialization.

Proof of Claim. Let $\mathcal{U}, \mathcal{U}'$ be two coverings of S accompanied by trivializations $\{\phi_i: y|_{U_i} \rightarrow f_i^*x\}_{\mathcal{U}}$ and $\{\gamma_j: y|_{U'_j} \rightarrow g_j^*x\}_{\mathcal{U}'}$. Refine the cover to

$$\mathcal{U}'' = \{U^{ij} := U_i \cap U'_j: \forall U_i \in \mathcal{U} \text{ \& } U'_j \in \mathcal{U}'\}.$$

Then the $\gamma_{ij} \circ (\phi_{ij})^{-1}: (f_{ij})^*x \rightarrow (g_{ij})^*x$ correspond to $h_{ij}: U_{ij} \rightarrow \Gamma$ inducing a family of isomorphisms, $(h_{ij})^*\alpha: (f_{ij})^*\mathcal{F}_{x,S} \rightarrow (g_{ij})^*\mathcal{F}_{x,S}$. So by descent of sheaves we have an isomorphism

$$h: \mathcal{F}_{y,T} \rightarrow \mathcal{F}'_{y,T}.$$

□ Claim

To define $\mathcal{F}$ on morphisms note that any morphism $H: y \rightarrow y'$ over $h: T \rightarrow T'$ in $\mathfrak{X}$ decomposes as $y \rightarrow h^*y' \rightarrow y'$. Then $\mathcal{F}(h^*y' \rightarrow y')$ is just the inverse of the adjunction $h^*\mathcal{F}_{y',T'} \rightarrow \mathcal{F}_{y',T'}$. We construct the map $\mathcal{F}(y \rightarrow h^*y')$ as follows. Let $\{U'_j\}$ be a cover of T' with a collection of trivializations $\phi'_j: y'|_{U'_j} \rightarrow (f'_j)^*x$. We can refine the $h^{-1}(U'_j)$ on T to a cover $\{U_i\}$ along with trivializations

$\phi_i: y|_{U_i} \rightarrow f_i^*x$. Let $h(U_i) \subseteq U'_{j(i)}$, our isomorphism restricts to $y|_{U_i} \rightarrow f^*y'|_{U'_{j(i)}|_{U_i}}$. Define

$$\gamma_i: f_i^*x \xrightarrow{(\phi_i)^{-1}} y|_{U_i} \xrightarrow{H} h^*(y'|_{U'_{j(i)}})|_{U_i} \xrightarrow{h|_{U_i}(\phi'_{j(i)})} h^*(f'_{j(i)})^*x|_{U_i} = (f'_{j(i)} \circ h|_{U_i})^*x$$

This is an element of $\underline{Isom}(x, x)$, so by the pre-stack condition corresponds to a map $g_i: U_i \rightarrow \Gamma$. Pulling α back along g_i we get an isomorphism,

$$\theta_i: f_i^*\mathcal{F}_{S,x} \xrightarrow{\sim} (f'_{j(i)} \circ h|_{U_i})^*\mathcal{F}_{S,x} \cong h^*\mathcal{F}_{S,x}|_{U_i}.$$

Once again, since α satisfies the cocycle condition, so do the θ_i on the overlaps. So by descent of sheaves, we obtain a map,

$$\mathcal{F}_{y,T} \rightarrow h^*\mathcal{F}_{y',T'}.$$

So precomposing the inverse of this map with the inverse of the adjunction map we get a morphism of sheaves, $\mathcal{F}_{y',T'} \rightarrow \mathcal{F}_{y,T}$ and evaluating this on the global sections we get our $\mathcal{F}(H)$. Since we are working with sheaves of sets, descent is unique and hence our constructed map is completely unique which implies our map $\mathcal{F}$ is a functor. Moreover, since for any $y/T, U \mapsto \mathcal{F}(y|_U) = \mathcal{F}_{y,T}(U)$ is a sheaf, $\mathcal{F}$ is a sheaf. Finally, $F(\mathcal{F}) = (\mathcal{F}_{x,S}, \alpha)$ since $\mathcal{F}(\phi|_U) = g^*(\alpha)$ where $g: U \rightarrow \Gamma$ is the map corresponding to $\phi|_U$. Therefore, F is essentially surjective.

The functor F is fully faithful: We need to show that for any sheaves $\mathcal{F}, \mathcal{G} \in \mathbf{Sh}(\mathfrak{X})$, the functor F induces a bijection

$$\mathrm{Hom}_{\mathbf{Sh}(\mathfrak{X})}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Sh}^\Gamma(S)}(F(\mathcal{F}), F(\mathcal{G}))$$

Let $F(\mathcal{F}) = (\mathcal{F}_{S,x}, \alpha)$ and $F(\mathcal{G}) = (\mathcal{G}_{S,x}, \beta)$. First we prove faithfulness. Let $\eta, \mu: \mathcal{F} \Rightarrow \mathcal{G}$ such that $\eta_x = \mu_x$. Let y/T be a family. Since x is versal there exists a covering $\{U_i\}$ of T and maps $f_i: U_i \rightarrow S$ and isomorphisms $\phi_i: y|_{U_i} \rightarrow f_i^* x$. But η and μ are natural so $\eta_{f_i^* x} = (f_i)^* \eta_x$ and likewise $\mu_{f_i^* x} = f_i^* \mu_x$. So using the naturality square for η (resp. μ which is not drawn) since $f_i^* \eta_x = f_i^* \mu_x$ and the vertical arrows in

$$\begin{array}{ccc} \mathcal{F}(y|_{U_i}) & \xrightarrow{\eta_{y|_{U_i}}} & \mathcal{G}(y|_{U_i}) \\ \mathcal{F}(\phi_i) \downarrow & & \downarrow \mathcal{G}(\phi_i) \\ \mathcal{F}(f_i^* x) & \xrightarrow{f_i^* \eta_x} & \mathcal{G}(f_i^* x) \end{array}$$

are isomorphisms we conclude conclude that

$$\eta_{y|_{U_i}} = \mu_{y|_{U_i}}$$

for all i . By descent of sheaves (which for sheaves of sets is unique), we conclude that $\eta_y = \mu_y$.

Finally we prove fullness. Let $\psi: (\mathcal{F}_{S,x}, \alpha) \rightarrow (\mathcal{G}_{S,x}, \beta)$ be a morphism of Γ -equivariant sheaves. We must construct a natural transformation $\eta: \mathcal{F} \Rightarrow \mathcal{G}$ such that $F(\eta) = \psi$.

Let y/T be a family in $\mathfrak{X}$ and, as usual, choose a trivialization $\phi_i: y|_{U_i} \rightarrow f_i^* x$. On each U_i , we define a local morphism $\eta_i: \mathcal{F}_{y,T}|_{U_i} \rightarrow \mathcal{G}_{y,T}|_{U_i}$ by pulling back ψ

$$\eta_i := f_i^* \psi: f_i^* \mathcal{F}_{S,x} \rightarrow f_i^* \mathcal{G}_{S,x}.$$

To show these glue to a unique $\mathcal{F}(y) \rightarrow \mathcal{G}(y)$ we must show these maps agree on the overlaps U_{ij} . As before, we have some transition maps $f_i^* x|_{U_{ij}} \rightarrow f_j^* x|_{U_{ij}}$ which correspond to $g_{ij}: U_{ij} \rightarrow \Gamma$ by the pre-stack condition. Pulling the equivariance condition $\beta \circ s^* \psi = t^* \psi \circ \alpha$ back along the g_{ij} produces equalities $g_{ij}^* \beta \circ f_i^* \psi = f_j^* \psi \circ g_{ij}^* \alpha$. But, as in the proof of essential surjectivity, $g_{ij}^* \alpha$ and $g_{ij}^* \beta$ are the transition maps $\theta_{ij}^{\mathcal{F}}$ of $\mathcal{F}_{y,T}$ and $\theta_{ij}^{\mathcal{G}}$ of $\mathcal{G}_{y,T}$. So

$$\begin{array}{ccc} f_i^* \mathcal{F}_{S,x} & \xrightarrow{\eta_i} & f_i^* \mathcal{G}_{S,x} \\ \theta_{ij}^{\mathcal{F}} \downarrow & & \downarrow \theta_{ij}^{\mathcal{G}} \\ f_j^* \mathcal{F}_{S,x} & \xrightarrow{\eta_j} & f_j^* \mathcal{G}_{S,x} \end{array}$$

commutes. But then by descent, η_i and η_j agree on the overlap. So again by descent these glue to a unique map η_y . By a similar argument as performed for $\mathcal{F}_{y,T}$, this map is independent of the choice of trivialization. Naturality of η follows when we decompose a morphism $H: y \rightarrow y'$ over h as a composite $y \rightarrow h^* y' \rightarrow y'$. $\square$

Remark 2.3.13. We can directly deduce that the definitions of constant, locally constant and constructible sheaves on sheaves are natural extensions of the definitions for topological spaces from Γ -equivariant descent. We deduce that $\mathbf{Sh}(X) \cong \mathbf{Sh}(\underline{X})$ by descent of Γ -equivariant sheaves. Note that $\underline{X}$ has some versal family x/X . Then $\underline{X} \xrightarrow{\text{id}} \underline{X}$ is the atlas with symmetry groupoid $X \rightrightarrows X$. We have an isomorphism between the categories of constant sheaves $\mathbf{Sh}^{\text{const}}(X) \simeq \mathbf{Sh}^{\text{const}}(\underline{X})$. Similarly, we have an equivalence between the categories of locally constant sheaves. In fact there is an equivalence of categories

$\mathbf{Sh}_p^{\text{const}}(\underline{X}, \mathcal{C}) \xrightarrow{\sim} \mathbf{Sh}_p^{\text{const}}(X; \mathcal{C})$. Since under the 2-Yoneda lemma, the collection $\{i_p : X_p \hookrightarrow X\}$ corresponds to $\{i_p : X_p \hookrightarrow \underline{X}\}$ and $(U \hookrightarrow X_p) \mapsto \mathcal{F}(i_p^* X)$ on $\underline{X}_p$ corresponds to the sheaf locally constant sheaf $\mathcal{F}|_{\underline{X}_p}$ on $\underline{X}_p$. $\diamond$

Lemma 2.3.14. Let $(\mathfrak{X}, P)$ be a stratified Deligne–Mumford stack. Then the sheaf $\mathcal{F} : \mathfrak{X}^{\text{op}} \rightarrow \mathbf{Set}$ is constructible if and only if for every family y/T the sheaf $\mathcal{F}_{y,T} \in \text{Ob}(\mathbf{Sh}_p(T))$ is constructible under the stratification

$$\underline{T} \xrightarrow{y} \mathfrak{X} \xrightarrow{s_{\mathfrak{X}}} \underline{P}.$$

Proof. If $\mathcal{F} : \mathfrak{X}^{\text{op}} \rightarrow \mathbf{Set}$ is constructible, then for every stratum $\mathfrak{X}_p \xrightarrow{i_p} \mathfrak{X}$, the sheaf $(i_p)^* \mathcal{F}$ is locally constant.

Let $y : T \rightarrow \mathfrak{X}$ be a family. We want to show that $y^* \mathcal{F}$ is constructible. Consider the p -th stratum. We have a map $y|_{T_p} : T_p \rightarrow \mathfrak{X}_p$. So it suffices to show that $y^* \mathcal{F}|_{T_p} = \left(y|_{T_p}\right)^* \circ i_p^* \mathcal{F}$ is locally constant.

Since $i_p^* \mathcal{F}$ is locally constant, there is a covering $\{U_i \xrightarrow{h_i} \mathfrak{X}_p\}$ in the étalé site $\mathcal{LH}(\mathfrak{X}_p)$ such that $h_i^* i_p^* \mathcal{F}$ is constant. Take the pullback,

$$\begin{array}{ccc} T_p \times_{\mathfrak{X}_p} U_i & \xrightarrow{y'} & U_i \\ h_i' \downarrow & \lrcorner & \downarrow h_i \\ T_p & \xrightarrow{y|_{T_p}} & \mathfrak{X}_p. \end{array}$$

Then $h_i^* (y^* \mathcal{F}|_{T_p}) \simeq y'^* (h_i^* i_p^* \mathcal{F})$ but the pullback of a constant sheaf is constant so $h_i^* (y^* \mathcal{F}|_{T_p})$ is constant. Since local homeomorphisms and effective epimorphisms are stable under base change, the collection $\{h_i' : T_p \times_{\mathfrak{X}_p} U_i \rightarrow T_p\}$ is a covering in $\mathcal{LH}(T_p)$. So $y^* \mathcal{F}|_{T_p}$ is locally constant. Therefore, by the previous remark, $\mathcal{F}_{y,S}$ is a constructible sheaf on S .

Conversely, if every $\mathcal{F}_{y,S}$ is constructible, then for a covering family $\{U_i \xrightarrow{h_i} \mathfrak{X}\}_i$ the sheaves $\mathcal{F}_{h_i, U_i}$ which are the $\mathcal{F}|_{U_i}$ are constructible for all i . But then since $\mathcal{F}|_{(U_i)_p}$ is locally constant for all p and we have a covering family $\{(U_i)_p \rightarrow \mathfrak{X}_p\}_i$ of $\mathfrak{X}_p$ the sheaf $\mathcal{F}|_{\mathfrak{X}_p}$ is locally constant (since each $\mathcal{F}|_{(U_i)_p}$ is locally constant, we can refine our covering of $\mathfrak{X}_p$ to some $\{V_i \rightarrow \mathfrak{X}_p\}$ such that $\mathcal{F}|_{V_i}$ is constant), completing the proof. $\square$

So we have a definition for constructible sheaves in the sense of definition 2.3.2, we stick to this definition if the stack is not Deligne–Mumford.

Definition 2.3.15 (Constructible Γ -equivariant sheaf). Let $(\mathfrak{X}, P)$ be a stratified topological stack with versal family x/S and symmetry groupoid $\Gamma \rightrightarrows S$. A Γ -equivariant sheaf $(\mathcal{F}, \phi)$ is **constructible** if $\mathcal{F}$ is constructible as a sheaf on S under the stratification $\underline{S} \xrightarrow{x} \mathfrak{X} \xrightarrow{s_{\mathfrak{X}}} \underline{P}$. $\diamond$

Corollary 2.3.16. Let $(\mathfrak{X}, P)$ be a stratified topological stack with versal family x/S and symmetry groupoid $\Gamma \rightrightarrows S$. Then there is an equivalence of categories,

$$\mathbf{Sh}_p^{\text{cons}}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Sh}_{\Gamma, P}^{\text{cons}}(S).$$

Proof. The Γ -equivariant descent holds. So we only need to show that the two notions of constructibility equate under this equivalence. If $\mathcal{F}$ is constructible then so is $x^*\mathcal{F}$ since x is a stratified map. So $(\mathcal{F}_{x,S}, \phi)$ is constructible.

Conversely, suppose $\mathcal{F}_{x,S}$ is constructible. We claim that $\mathcal{F}_{y,T}$ is constructible. Let $\{U_i\}$ be a trivialization of some y/S . Refine it so that $f_i^*\left(\mathcal{F}_{x,S}|_{X_p}\right)$ is constant. Then since,

$$i_p^*\left(\mathcal{F}_{y,T}|_{X_p \cap U_i}\right) \cong f_i^*\left(\mathcal{F}_{x,S}|_{X_p}\right)$$

each $i_p^*\mathcal{F}_{y,T}$ is locally constant, so $\mathcal{F}$ is constructible. $\square$

Suppose that the stack is moreover conically stratified. Under the exodromy correspondence, a Γ -equivariant constructible sheaf $(\mathcal{F}, \phi)$ on S corresponds to a pair $(\rho_{\mathcal{F}}, \rho(\phi))$ such that

$$\rho(\phi)(g) \circ \rho(\phi)(f) = \rho(\phi)(g * f).$$

Reformulating this data in terms of a pair in a functor category of an object and morphism, we obtain the following definition of Γ -equivariant exodromy representation.

Definition 2.3.17. Let $\Gamma \rightrightarrows S$ be the symmetry groupoid of a topological stack $\mathfrak{X}$ as above. A **Γ -equivariant representation** is a pair (F, η) consisting of a functor $F: \Pi_1^{\text{strat}}(S) \rightarrow \mathbf{Set}$ and a natural isomorphism $\alpha: F \circ s_* \xrightarrow{\sim} F \circ t_*$ such that for all $f, g \in \Gamma$ composable, $\alpha(g) \circ \alpha(f) = \alpha(g * f)$. A morphism $\eta: (F, \alpha) \rightarrow (G, \beta)$ is a natural transformation $\eta: F \Rightarrow G$ such that

$$\begin{array}{ccc} F \circ s_* & \xrightarrow{\alpha} & F \circ t_* \\ \eta \circ s_* \downarrow & & \downarrow \eta \circ t_* \\ G \circ s_* & \xrightarrow{\beta} & G \circ t_* \end{array}$$

commutes on the nose. We write $\mathbf{Fun}_{\Gamma}(\Pi_1^{\text{strat}}(S), \mathbf{Set})$ for the category of Γ -equivariant representations. $\diamond$

Lemma 2.3.18. If $\mathfrak{X}$ is conically stratified, we may apply the ordinary exodromy correspondence for topological spaces to the category of Γ -equivariant sheaves on S to obtain an equivalence of categories,

$$\mathbf{Sh}_{\Gamma, P}^{\text{cons}}(S) \rightarrow \mathbf{Fun}_{\Gamma}(\Pi_1^{\text{strat}}(S), \mathbf{Set}), \quad (\mathcal{F}, \phi) \rightarrow (\rho_{\mathcal{F}}, \rho(\phi)).$$

Proof. The essential image of the Γ -equivariant sheaves under the exodromy correspondence (the fiber functor followed by the sheaf of sections functor) are the Γ -equivariant representations. $\square$

2.3.4 Exodromy for Conically Stratified Topological Stacks

The descent results also imply a stacky version of exodromy. In this section we formulate it. In the terminology of [Jan24], we would describe the symmetry groupoid of a stack $\mathfrak{X}$ as the 2-Čech nerve of $\mathfrak{X}$. The exit-path category of the stack is then shown to be equivalent to the 2-coequalizer of this nerve under the pushforward functor, [Jan24, Corollary 3.17], requiring the additional hypothesis that the poset satisfies the ascending chain condition. So we might as well take this for our definition of the exit-path category. We want descent over closed covers to hold for our construction to work. For this it is sufficient to assume $\mathfrak{X}$ is conically stratified.

Definition 2.3.19. Let $(\mathfrak{X}, P)$ be a conically stratified stack and suppose moreover that P satisfies the ascending chain condition. The **exit-path category** of $\mathfrak{X}$ written $\Pi_1^{\text{Strat}}(\mathfrak{X}, P)$ is the 2-coequalizer of

$$\Pi_1^{\text{Strat}}(\Gamma) \rightrightarrows \Pi_1^{\text{Strat}}(S). \quad \diamond$$

Then, from our previous discussion of Γ -equivariant sheaves, we obtain the following result.

Theorem 2.3.20 (Exodromy for Conically Stratified Topological Stacks). Let $(\mathfrak{X}, P)$ be a conically stratified topological stack such that P satisfies the ascending chain condition. Then, via Γ -equivariant descent for constructible sheaves, we obtain an exodromy correspondence for conically stratified topological stacks.

Proof (Sketch). The theorem is a direct consequence of Γ -equivariant descent and exodromy for topological spaces. Once we show that the exodromy correspondence commutes with this descent. $\square$

2.3.5 The Proper Base Change Theorem for Topological Stacks

In this section we simply state and prove the proper base change theorem for topological stacks for the case $i = 1$. The proof follows directly from the proper base change theorem for topological spaces by Γ -equivariant descent.

Theorem 2.3.21 (The Proper Base Change Theorem for Topological Stacks). Consider the cartesian square

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{q} & \mathcal{Y} \\ \pi' \downarrow & \lrcorner & \downarrow \pi \\ \mathfrak{X}' & \xrightarrow{p} & \mathfrak{X} \end{array}$$

of topological stacks. If π is a proper morphism of stacks then for any sheaf $\mathcal{F}$ on $\mathcal{Y}$ we have an isomorphism

$$p^* \mathbf{R}^1 \pi_* \mathcal{F} \xrightarrow{\sim} \mathbf{R}^1 \pi'_* q^* \mathcal{F}.$$

Proof (Sketch). Apply Γ -equivariant descent. Since the map resulting from π is a proper map of topological spaces, we may apply the proper base change theorem. Another application of Γ -equivariant descent to our obtained isomorphism of sheaves gives us our desired isomorphism of sheaves of stacks, completing the proof. $\square$

Chapter 3

Exodromy for Moduli of Stable Curves

‘And I spent my life
Watching it go by from the sidelines
And God, I’ve tried
But I think it’s about time I put a fight’
🎵 *Sun Bleached Flies* — Ethel Cain

In this chapter we discuss the constructions considered Jansen’s paper where the exit path category for the moduli of stable curves is computed in the more general context of ∞ -categories [Jan23]. This decomposition turns out to be well-suited for computing the exodromy representations.

3.1 The Stratification on the Moduli of Stable Curves

In this section we define the stratification endowed on the moduli of stable curves. To every stable nodal curve/point on the stack we can associate a graph. Ordering these graphs according to specializations we obtain a poset. Extending this construction to families, we obtain a stratification for the stack. First we define the notion of dual graph, taken almost verbatim from [Jan23, p. 36].

Definition 3.1.1 (Dual Graph). An n -marked dual graph is a tuple $(\mathbf{G}, \omega, m)$ where

- $\mathbf{G}$ is a **graph** which is a tuple (V, E, ι) consisting of
 - A set of **vertices** V .
 - A set of **half-edges** $E = \bigsqcup_{v \in V} E_v$ partitioned according to V .
 - An involution $\iota: E \rightarrow E$.
- A weight function $\omega: V \rightarrow \mathbb{Z}_{\geq 0}, v \mapsto g_v$.
- A bijection $m: [n] \rightarrow E$ labelling the **markings** of the graph which are the fixed points of the involution.

The **genus** of an n -marked dual graph (G, ω, m) is given by $g(\mathbf{G}, \omega, m) = 1 - \chi(\mathbf{G}) + \sum_{v \in V} \omega(v)$. Where $\chi(G)$ is the Euler characteristic of a graph. A dual graph is stable if for every $v \in V$

$$2\omega(v) - 2 + |E_v| > 0.$$

Two n -marked dual graphs are isomorphic if they are isomorphic as graphs and the isomorphism preserves the weight functions and the markings. $\diamond$

Every nodal curve has an associated dual graph.

Definition 3.1.2 (Dual Graph Associated to Nodal Curve). [Jan23, Construction A.6] Let $(X; p_i)$ be a stable nodal curve over $\mathbb{C}$ with normalization $\tilde{X} \rightarrow X$. The associated **dual graph** $G(X; p_i)$ is the graph (G, ω, m) defined as follows:

- The vertices V are the connected components of $\tilde{X}$.
- The set of half-edges $E = \coprod_{v \in V} P_v$ where P_v is the set of marked points or nodes on component v .
- The involution of E , $\iota: E \rightarrow E$ is given by

$$e \mapsto \begin{cases} e & e \text{ is a marked point on } X \\ e' & e, e' \text{ correspond to the same node on } X. \end{cases}$$

- The morphism $\omega: V \rightarrow \mathbb{Z}_{\geq 0}$ maps a vertex v to the genus $g(\tilde{X}_v)$ of the corresponding component $\tilde{X}_v \subseteq \tilde{X}$.
- The markings are indexed by $m: P \rightarrow E, i \mapsto p_i$. $\diamond$

Under this definition of the dual graph, there are definitions of genus and stability for stable graphs and these coincide with those for stable nodal curves [ACG11, Chapter X, Section 2 and Lemma 3.1]. Now we define what a specialization is.

Definition 3.1.3 (Contraction, [Jan23, Construction A.2]). Let $\mathbf{G} = (G, \omega, m)$ be an n -pointed dual graph, where $G = (V, E, \iota)$ consists of vertices V , half-edges $E = \coprod_V E_v$, and an involution $\iota: E \rightarrow E$. Let $E' \subseteq E$ be an ι -invariant subset containing no markings. Then the **contraction of $\mathbf{G}$ along E'** is the graph $\mathbf{G}/E' = (G', \omega', m')$ defined by collapsing the edges of E' :

- The vertices V' of G' are the connected components of the subgraph $(V, E', \iota|_{E'})$.
- The half-edges are $E \setminus E'$, with involution $\iota|_{E \setminus E'}$ and where a half-edge e' is incident to vertex v' if it is contained in $E'_{v'} = \coprod_{v \in v'} (E_v \setminus E')$.
- The weight function is $\omega': V \rightarrow \mathbb{Z}_{\geq 0}$ where $\omega'(v)$ is the genus of the connected component corresponding to v in the subgraph $(V, E', \iota|_{E'})$ of G .
- The marking m' is the factorization of m through $E \setminus E' \subset E$. $\diamond$

Definition 3.1.4 (Specialization). We say that $\mathbf{G}'$ is a specialization of $\mathbf{G}$, written $[\mathbf{G}'] \leq [\mathbf{G}]$, if $\mathbf{G}$ is isomorphic to a contraction of $\mathbf{G}'$. This defines a partial order on the isomorphism classes of stable n -marked genus g stable dual graphs. The poset is written $\mathcal{G}_{g,n}$. $\diamond$

The moduli stack of curves is stratified by the poset of specializations of n -marked stable curves $\mathcal{G}_{g,n}$. Note that the graph associated to the smooth curve of type (g, n) is the largest element of this poset. By [Jan23, Lemma 2.9], the topological space $\overline{M}_{g,n}^{\text{top}}$ is $\mathcal{G}_{g,n}$ -stratified under the map

$$\pi_{g,n}: \overline{M}_{g,n}^{\text{top}} \rightarrow \mathcal{G}_{g,n}, \quad [(C; p_i)] \mapsto [\mathbf{G}(C; p_i)].$$

The moduli stack inherits this stratification under the coarse moduli map,

$$\pi: \overline{\mathcal{M}}_{g,n}^{\text{top}} \xrightarrow{\text{coarse moduli}} \overline{M}_{g,n}^{\text{top}} \xrightarrow{\pi_{g,n}} \mathcal{G}_{g,n}.$$

3.2 The Charney-Lee Category of Admissible Curves

In this section we construct the Charney-Lee category associated to a stable nodal surface and its admissible curve systems.

3.2.1 Stable Nodal Surfaces and the Charney-Lee Category

These notes follow section 2 in [Jan23] closely.

Definition 3.2.1. A **surface** is a real oriented smooth 2-manifold. A **P -pointed surface of finite type** (S, P) is a compact connected surface S together with a finite set P of marked points and an embedding $P \hookrightarrow S$. We say that such a surface is **stable** if $2g(S) - 2 + \#P > 0$. $\diamond$

Definition 3.2.2 ([FM11, Section 1.2.5, 1.2.6]). Let S be a surface. A morphism $H: S^1 \times I \rightarrow S$ is an **isotopy** from γ to γ' if it is a homotopy from γ to γ' such that for every $t \in I$ the closed curve $H(S^1, t)$ is simple, meaning that that map $S^1 \rightarrow S$ is injective. An **isotopy of S** is a homotopy $H: S \times I \rightarrow S$ such that for every $t \in I$, the map restricted to $S \times \{t\}$ is a homotopy. $\diamond$

Definition 3.2.3 ([Jan23, Definition 3.1]). A **topological nodal surface of finite type** is an $S \in \mathbf{Top}$ such that every $s \in S$ has a fundamental system of neighborhoods (f.s.o.n.) such that either

1. all neighborhoods are homeomorphic to the unit disc in $\mathbb{R}^2$
2. all neighborhoods are homeomorphic to the cone $C(S^1)$ in $\mathbb{R}^3$. $\diamond$

Definition 3.2.4 ([Jan23, Definition 3.4]). A **stable nodal surface** is a P -pointed topological nodal surface of finite type (S, P) along with a smooth structure on each component of the normalization (S_v, P_v) and such that each (S_v, P_v) is a stable topological surface. $\diamond$

Definition 3.2.5 ([Jan23, p.12]). The **mapping class group** of a stable nodal surface (S, P) is the group $\Gamma_S = \Gamma_{(S,P)} = \pi_0(\text{Diff}_P^+(S)) \cong \pi_0(\text{Homeo}_P^+(S))$ of isotopy classes of orientation preserving P -pointed diffeomorphisms of stable surfaces.

The subgroup $\tilde{\Gamma}_S \subseteq \Gamma_S$ of those automorphisms preserving the components and fixing the nodes is called the **pure mapping class group**. $\diamond$

Remark 3.2.6 ([Jan23, Observation 3.6]). If (S, P) is non-singular then $\tilde{\Gamma}_S = \Gamma_S$. In general,

$$\tilde{\Gamma}_S = \prod \Gamma_{S_v}.$$

◇

There is a canonical way to associate a stable graph to a stable nodal surface. This is performed similarly to the construction of the dual graph in definition 3.1.2, therefore we omit this construction (see [Jan23, Construction 3.7]).

Definition 3.2.7 ([Jan23, Definition 3.8]). The **genus** of a nodal surface (S, P) is the genus of its dual graph $G(S, P)$. ◇

Lemma 3.2.8 ([Jan23, Lemma 3.9]). Let (S, P) be a stable nodal surface, then

$$1 \rightarrow \tilde{\Gamma}_S \rightarrow \Gamma_S \rightarrow \text{Aut}(G(S, P)) \rightarrow 1$$

is exact.

We discuss the construction of obtaining a nodal surface from a nodal curve. Let $(C; p_i)$ be a stable nodal curve. Take its normalization $\nu: \tilde{C} \rightarrow C$. The normalization is a disjoint union of smooth curves, which inherits the marking p_i but additionally for every node c_i two markings $\{c_i, c'_i\} = \nu^{-1}(c_i)$. After analytification we obtain a disjoint union of Riemann surfaces $(\tilde{C})^{\text{an}}$ inheriting the markings p_i and c_i, c'_i . By gluing the c_i and c'_i together and setting $P = \{p_i\}$ we obtain a stable nodal surface $(S_C; p_i)$.

3.2.2 Admissible Curve Systems

Definition 3.2.9. A **simple closed curve** in some topological surface S is a morphism $\gamma: S^1 \rightarrow S$ which is a closed embedding. ◇

A simple closed curve γ bounds a disc if there exists a topological embedding $\Gamma: D^2 \rightarrow S$ such that $\Gamma(\partial D^2) = \gamma(S^1)$. We say that a simple closed curve γ bounds a disc once-punctured by either a marking or a node, if there exists a topological embedding $\Gamma': D^2 \setminus \{0\} \rightarrow S$ such that $\Gamma'(\partial D^2) = \gamma(S^1)$ and such that the puncture is a marking or a node.

A pair of simple closed curves γ and γ' are said to bound a cylinder if there exists a topological embedding $\Gamma: S^1 \times [0, 1] \rightarrow S$ such that $\Gamma(S^1 \times \{0\}) = \gamma(S^1)$ and $\Gamma(S^1 \times \{1\}) = \gamma'(S^1)$.

Definition 3.2.10. An **admissible curve system** on a stable nodal surface (S, P) is a disjoint collection C of simple closed curves in $S \setminus \{\text{Nodes} \cup P\}$ such that:

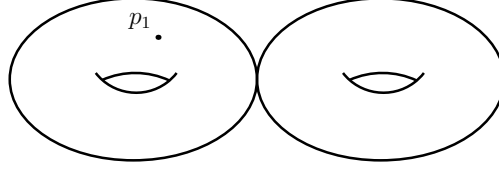
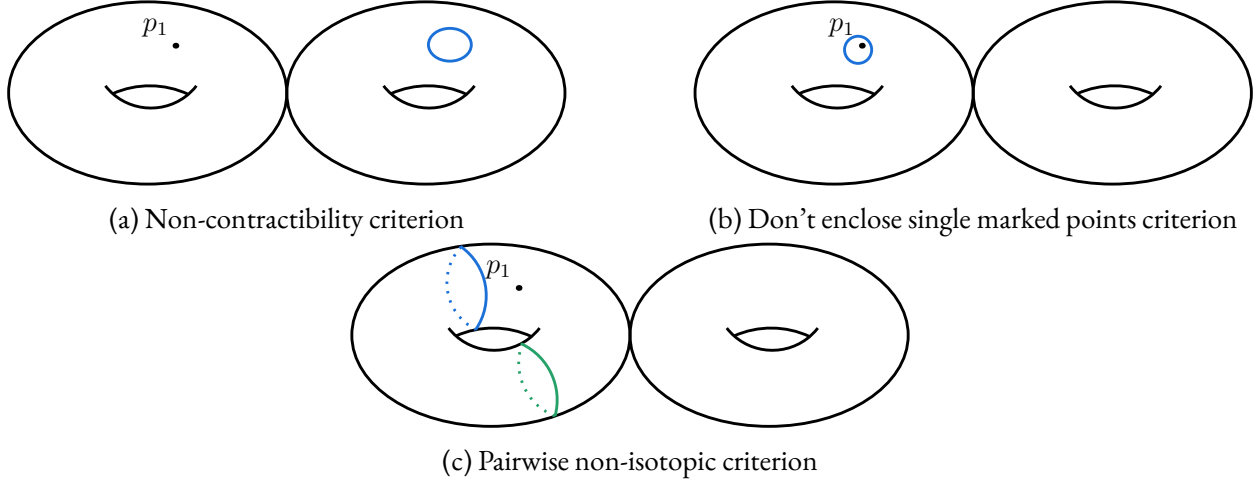
Non-contractibility: no curve bounds a disc;

No singling out marked points: no curve bounds a disc which is once-punctured by either a node or a marking;

Pairwise non-isotopic no pair of curves bounds a cylinder. ◇

Example 3.2.11 (The admissible curves of a nodal stable surface.). Consider the nodal stable surface S_1 depicted in fig. 3.1.

We can illustrate each of the three criteria in action on S_1 . Figure 3.2 depicts three systems of closed curves each breaking one of the criteria. ◇

Figure 3.1: The nodal stable surface S_1 with marked point p_1 .Figure 3.2: Examples of breaking the three criteria for admissible curve systems on S_1 . (a) A contractible curve. (b) A curve containing a single marked point. (c) Two isotopic curves.

Definition 3.2.12. Let (S, P) be a stable nodal surface. Define $\mathfrak{C}_S$ to be the partially ordered set of isotopy classes of admissible curve systems on (S, P) where

$$\sigma \leq \tau \iff \tau \subset \sigma$$

where we say τ is contained in σ if each curve in τ is isotopic on (S, P) to a closed curve in σ . By convention the empty admissible system $\emptyset$ is contained in $\mathfrak{C}_S$. $\diamond$

Definition 3.2.13 (Quotienting by an admissible system of curves). Given an admissible curve system C on a stable nodal surface (S, P) , there is a stable nodal surface $(S/C, P)$ defined as follows:

- S/C is the topological surface obtained from S by contracting each simple closed curve of C to a node.
- In this case $P \hookrightarrow S \twoheadrightarrow S/C$ is still an embedding
- The resulting smooth components inherit their smooth structures from the original space (we omit the details of this construction; see [Jan23, Construction 3.12]). $\diamond$

Remark 3.2.14 (Motivating the single marked points condition). Imposing non-contractibility and pairwise non-isotopic criteria is understandable: we don't want an admissible curve system to contain redundant information. Constraining our study to those curves that do not contain a single marked point is necessary if we want the quotienting operation to preserve stability. Consider a 3-pointed Riemann

sphere and a loop enclosing one of the three marked points. Collapsing this curve produces a one-pointed Riemann sphere and a two-pointed Riemann sphere. Under the normalization, the component v corresponding to the one-pointed Riemann sphere contains two special points (the original marked point and the preimage of the new node). Consequently,

$$2 \cdot g(v) - 2 + n = 2 \cdot 0 - 2 + 2 = 0$$

so S/C is not stable. $\diamond$

Along with preserving stability, the quotient operation preserves the genus, that is, $g(S/C) = g(S)$. Moreover, $G(S/C; P)$ is a specialization of $G(S; P)$. For example, in the case of the two-marked torus and some chosen admissible system see fig. 3.3.

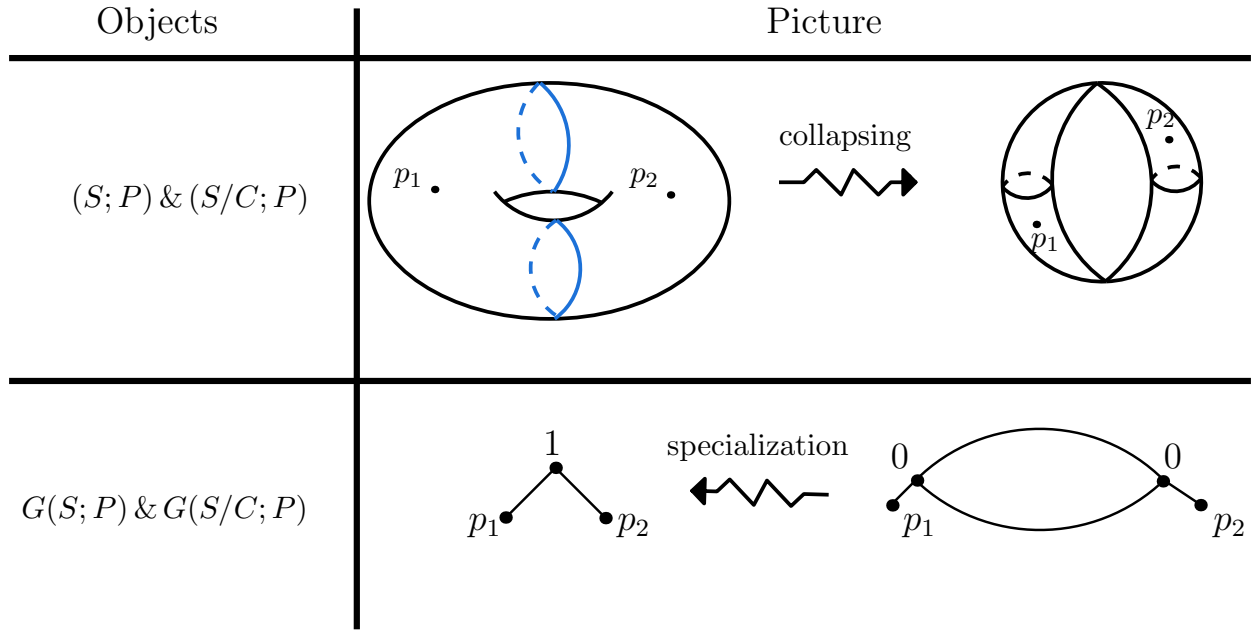


Figure 3.3: The quotient operation is reverse to the specialization order for the two-pointed torus with admissible curve system P depicted in blue above.

Looking back at the surface S_1 considered in example 3.2.11, if we were to compute $\mathcal{C}_S$ we would see that the viable admissible curves on each component are the same. So we can say something along the lines of

“ $\mathcal{C}_S$ does not see lone marked points.”

While it clearly does distinguish cases with multiple marked points as seen in fig. 3.3 where an admissible curve system that doesn't exist in either component of S^1 is considered. Let Γ_S be the mapping class group of S . These lone markings are recovered once we consider the action $\Gamma_S \curvearrowright \mathcal{C}_S$ defined as follows: Given an isotopy class of orientation-preserving automorphisms $\bar{\gamma}$ and an isotopy class of admissible curves $[C]$ define

$$\bar{\gamma} \cdot [C] := [\gamma(C)].$$

In fact, under this action, determining the orbit in which some admissible curve system lies is equivalent to determining the stratum (in $\overline{\mathcal{M}}_{G(S;P)}$) the corresponding nodal curve $\widehat{S/C}$ lies in:

Proposition 3.2.15. [Jan23, Proposition 3.14] The map

$$\Gamma_S \setminus \mathcal{C}_S \rightarrow \mathcal{G}_{G(S;P)}, \quad \overline{C} \mapsto G(S/C;P)$$

is an isomorphism of partially ordered sets.

So not only do the admissible systems “see the marked points” but we recover the specializations of the dual graph. We would like to say something along the lines of “admissible curve systems are thus formulated to precisely encapsulate the specialization poset of the dual graph, nothing more and nothing less.”

3.2.3 The Charney-Lee Category

For every stable nodal surface (S, P) there is a 1-category CL_S , its construction uses Dehn twists.

Dehn twists

Definition 3.2.16. Let (S, P) be a stable nodal surface and γ a simple closed curve in $S \setminus \{\text{nodes}\} \cup P$. Let $\varphi: A_\gamma \xrightarrow{\sim} S^1 \times I$ be a tubular neighborhood of γ with the local coordinates denoted (s, t) , where S^1 is parameterized by e , i.e. $s = e^{i\sigma}$, $\sigma \in [0, 2\pi]$. An (A_γ, φ) -**Dehn twist** is an orientation preserving diffeomorphism f on S such that

1. $f|_{S \setminus A_\gamma} = \text{id}$
2. On A_γ , under φ (that is, in terms of local coordinates (s, t)), f is the map $(s, t) \mapsto (se^{2\pi it}, t)$ $\diamond$

On [FM11, p. 68], it is stated that the isotopy class of the Dehn twist does not depend on the choice of trivialization φ . We will subsequently denote the Dehn twist associated to a closed curve γ , by δ_γ . Figure 3.4 illustrates how a Dehn twist deforms an s -contour line.

These are of interest to us as a result of this theorem due to Dehn:

Theorem 3.2.17 (Dehn-Lickorish theorem). [FM11, Theorem 4.1, Theorem 4.2, Theorem 5.7] Let S be an oriented smooth surface of genus g without markings. The mapping class group Γ_S is generated by finitely many Dehn twists (not true with markings, see [FM11, pp. 93-94]), has a solvable word problem (using the bigon criterion and Alexander method; also true if the surface has markings) and is finitely presentable.

For any $\sigma \in \mathcal{C}_S$ let $\Gamma_S(\sigma) \subseteq \Gamma_S$ be the subgroup of mapping classes preserving σ and let $\Delta_\sigma \subseteq \Gamma_S(\sigma)$ be the free abelian group generated by the Dehn twists of the closed curves in σ .

Remark 3.2.18. The Δ_σ have the following properties:

- Let $\tau, \sigma \in \text{Ob}(\mathcal{C}_S)$ such that $\tau \leq \sigma$ then $\Delta_\sigma \subseteq \Delta_\tau$ (since $\sigma \subseteq \tau$ implies that every Dehn twist δ_c for $c \in \sigma$ is equivalent to some δ_d for $d \in \tau$ in the mapping class group Γ_S ; [FM11, Fact 3.6]).
- Under the action $\Gamma_S \curvearrowright \mathcal{C}_S$, we have formula $\gamma \cdot \Delta_\sigma \cdot \gamma^{-1} = \Delta_{\gamma \cdot \sigma}$ [FM11, Fact 3.7]. $\diamond$

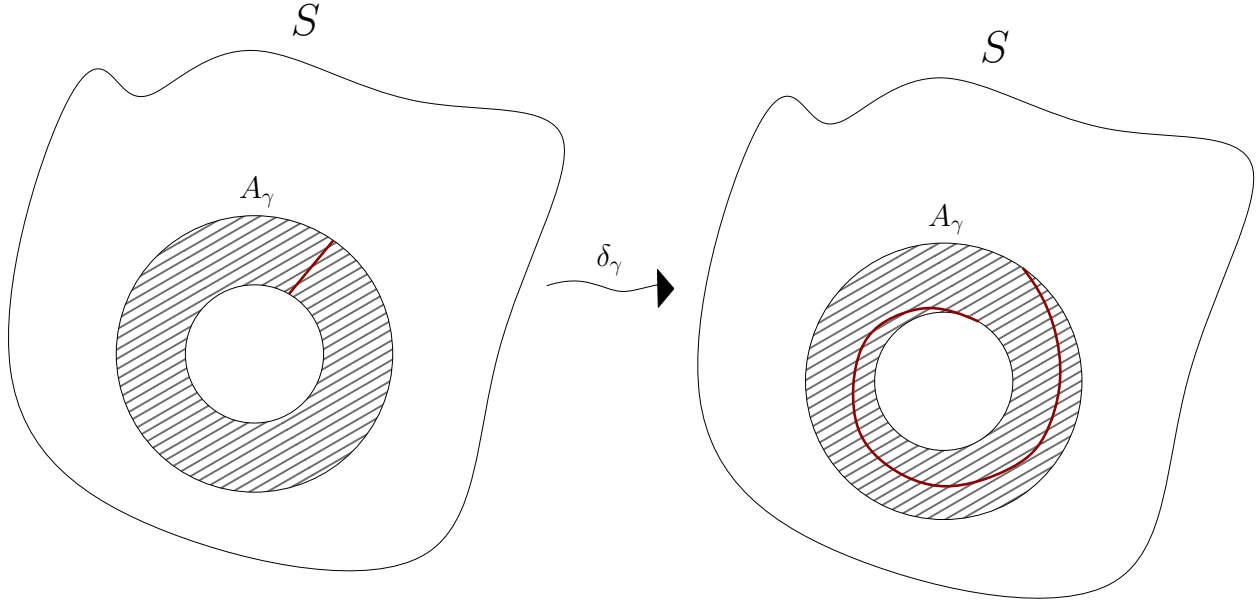


Figure 3.4: The Dehn twist associated to γ on nodal stable surface S twists the straight s -contours within tubular neighborhood A_γ . Figure drawn up by author in Ipe, a re-sketch of the first half of [FM11, Figure 3.1].

Defining CL_S

Definition 3.2.19. We define $\text{CL}_{(S,P)} = \text{CL}_S$ by

Objects: The objects are $\text{Ob}(\mathfrak{C}_S)$.

Morphisms: For two admissible curve systems σ, τ define

$$\text{Hom}_{\text{CL}_S}(\sigma, \tau) := \{\gamma \in \Gamma_{(S,P)} : \gamma \cdot \sigma \leq \tau\} / \Delta_\sigma.$$

Where composition of morphisms is given by composition in the mapping class group. $\diamond$

Remark 3.2.20 (Morphisms Characterize Generalizations of Dual Graphs). Let (S, P) be a stable nodal surface (see definition 3.2.4). For an admissible curve system σ , let $G(\sigma)$ be the dual graph associated to the nodal surface S/σ . Using this notation we are able to interpret the morphisms of CL_S . In fact, there exists a morphism $\gamma : \sigma \rightarrow \tau$ if and only if $G(\sigma)$ is a specialization of $G(\tau)$. Indeed, if such a morphism exists then the graph $G(\gamma \cdot \sigma) \cong G(\sigma)$ is a specialization of $G(\tau)$. Conversely by proposition 3.2.15, if $[G(\sigma)] \leq [G(\tau)]$ then $\bar{\sigma} \leq \bar{\tau}$ so there exists a $\gamma \in \Gamma_{(S,P)}$ such that $\gamma \cdot \sigma \leq \tau$. Observe that for any mapping class $\gamma \in \Gamma_{(S,P)}$ the dual graph $G(\sigma)$ and $G(\gamma \cdot \sigma)$ are isomorphic as abstract graphs, so such a morphism exists if and only if $G(\sigma)$ is a specialization of $G(\tau)$ determined by γ . Now note that $\gamma \circ \gamma'^{-1} \in \Delta_\sigma$ if and only if the graphs $G(\gamma \cdot \sigma)$, $G(\gamma' \cdot \sigma)$ are isomorphic as specializations of $G(\tau)$ by proposition 3.2.15. In particular, we conclude that $\text{Hom}_{\text{CL}_S}(\sigma, \tau)$ characterizes the distinct ways nodal graphs isomorphic to $G(\sigma)$ generalize to $G(\tau)$. $\diamond$

The isomorphism classes of objects of CL_S correspond to the strata of $\overline{\mathcal{M}}_{g,n}$. So by the previous remark CL_S is equivalent to a category whose objects are the strata of $\overline{\mathcal{M}}_{g,n}$ and whose morphisms $\sigma \rightarrow \tau$ are parameterized by the specializations of $G(\tau)$ (not considered up to isomorphism) isomorphic to $G(\sigma)$.

Remark 3.2.21. Interestingly, if (S, P) is a nonsingular stable surface then CL_S is equivalent to the opposite of the original category devised in the paper by Charney and Lee [CL84], denoted $\mathfrak{SC}_{g,P}^{\text{op}}$ in Jansen's paper (see [Jan23, Proposition 3.23]). $\diamond$

3.3 Computation of the Exit Path Category

Jansen defines (see [Jan24, Definition 3.12]) the exit path category of (X, P) as the opposite of the full subcategory generated by the atomic constructible sheaves. For the underlying 1-categories this means

$$\Pi(X, P) := (\mathbf{Sh}_P^{\text{cons}}(X)^{\text{atom}})^{\text{op}}.$$

In this section, we will discuss how Jansen decomposes the exit path category of $\overline{\mathcal{M}}_{g,n}^{\text{Top}}$ inductively in terms of locally closed subsets of some topological space $\overline{\mathcal{T}}_{g,n}$ called the bordified Teichmüller space. The exit path category of $\overline{\mathcal{T}}_{g,n}$ happens to coincide with its stratifying poset. In fact, Jansen shows that the exit path category of $\overline{\mathcal{M}}_{g,n}^{\text{Top}}$ is equivalent to the Charney-Lee category of the smooth n -marked genus g topological surface, which we denote by $S_{g,n}$.

3.3.1 The Harvey Bordification of the Teichmüller Space

The Harvey bordification of the Teichmüller space is defined in [Jan23, Section 4] and discussed in detail in [Iva90]. We briefly review its construction in the next section.

The Teichmüller Space Associated to a Stable Nodal Surface (S, P)

In this section we paraphrase [Jan23, Section 4.1].

Definition 3.3.1. A **stable Riemann surface with nodes** is a compact connected complex analytic space S such that $\forall x \in S$ there exists an open neighborhood U_x of x in S such that $U_x \cong \{|x| < 1\} \subset \mathbb{C}$ or $U_x \cong \{xy = 0\} \subset \mathbb{C}^2$ in which case x is a **node** of S . $\diamond$

Let (S, P) be a stable nodal surface. Using the notion of stable Riemann surfaces with nodes we define a manifold $\mathcal{T}_S$ called the **Teichmüller space** associated to (S, P) .

Definition 3.3.2. An **isometry** is a morphism of metric spaces which is distance preserving. $\diamond$

Definition 3.3.3. For a stable nodal surface (S, P) (which is really a curve considered as Riemann surface) a **marking of the curve** is a pair (X, f) consisting of a Riemann surface with nodes X and an orientation preserving homeomorphism

$$f: S \setminus P \rightarrow X.$$

Two markings (X, f) and (Y, g) are **equivalent** if there exists a $\gamma: S \setminus P \rightarrow S \setminus P$ which is an orientation preserving diffeomorphism isotopic to the identity and an isometry $\varphi: X \rightarrow Y$ such that

$$\begin{array}{ccc} S \setminus P & \xrightarrow{\gamma} & S \setminus P \\ f \downarrow & & \downarrow g \\ X & \xrightarrow{\varphi} & Y \end{array}$$

commutes. This turns out to be an equivalence relation and a class $[(X, f)]$ is called a **marked stable curve**. $\diamond$

A marking of the curve is essentially just a complex structure on $S \setminus P$ away from the nodes. To define the topology on this set, and for that matter the smooth structure, we note that a smooth structure on the stable nodal surface (S, P) can be reduced to a point in $(\mathbb{R}_{>0} \times \mathbb{R})^C$ where C is a maximal admissible curve system on S via something called the **Fenchel-Nielsen coordinates**. In particular, given a maximal admissible system $C = \{c_1, \dots, c_m\}$ on (S, P) , we have a pants decomposition of the surface S into pieces depicted in fig. 3.5.

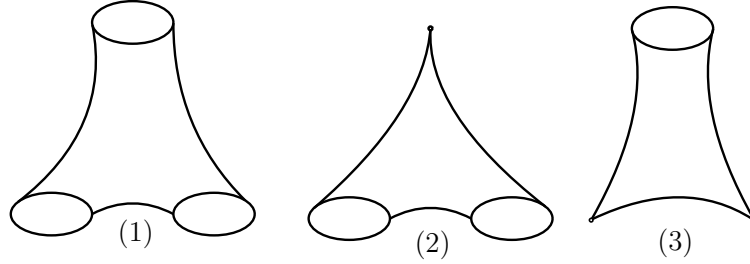
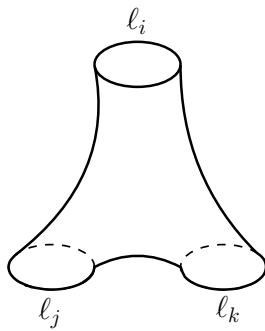


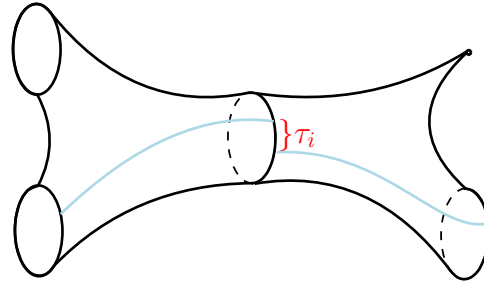
Figure 3.5: **(1)** Pants (a pair of Levi's). **(2)** A punctured annulus. **(3)** A twice punctured disc. A redrawing of [Har88, Figure 1.2] created by the author.

Theorem 3.3.4 (Fenchel-Nielsen coordinates). [Jan23, p. 19] Let S be a stable Riemann surface with nodes. Consider any maximal admissible curve system C . All hyperbolic structures on S can be built by gluing the pieces in the pants decomposition according to the Fenchel Nielsen coordinates, that is, we have a homeomorphism

$$\mathcal{T}_S \xrightarrow{\sim} \prod_C (\mathbb{R}_{>0} \times \mathbb{R}), [X, f] \mapsto (\ell_1, \dots, \ell_m, \tau_1, \dots, \tau_m).$$



(a) Coordinate ℓ_i is the length of closed curve C_i . Drawn up by the author.



(b) The twist coordinate $\tau_i \in \mathbb{R}$, defined as the signed hyperbolic distances between feet of the perpendicular feet (■) dropped on S (keep in mind it is not 2π -periodic). This figure was drawn up by the author directly from [Har88, Figure 1.3].

Figure 3.6: The Fenchel-Nielsen coordinates (ℓ_i, τ_i) parameterizing the Teichmüller space.

The geometrical interpretations of these coordinates are depicted in figs. 3.6a and 3.6b. Moreover, defining the topology in this way is independent of the choice of maximal curve system (up to unique isomorphism). $\diamond$

So $\mathcal{T}_S$ is a smooth manifold. We can define a group action $\Gamma_S \curvearrowright \mathcal{T}_S$ by,

$$\bar{\gamma} \cdot [X, f] := [X, f \circ \gamma^{-1}]$$

Lemma 3.3.5. [Jan23, Proposition 4.2, 4.4] The action of Γ_S on $\mathcal{T}_S$ is properly discontinuous. So the quotient stack is Deligne-Mumford. In fact the quotient stack $[\mathcal{T}_S/\Gamma_S]$ is isomorphic to the moduli stack $\mathcal{M}_{\mathbf{G}(S;P)}^{\text{top}}$.

So we have successfully defined the universal cover of $\mathcal{M}_{\mathbf{G}}$. We next attempt to do this for the Deligne-Mumford compactification.

The Harvey Bordification

In this subsection we define the Harvey bordification $\overline{\mathcal{T}}_S$ of a marked topological surface (S, P) .

Definition 3.3.6. Let (C, X, f) be a triple consisting of an admissible curve system C on stable nodal surface (S, P) , a stable nodal Riemann surface X and an orientation preserving homeomorphism away from the marking and curve system $f: S \setminus (C \cup P) \rightarrow X$. This triple is said to be a **singular hyperbolic structure** on (S, P) if for every curve c in the admissible system C there exists an open neighborhood U of c (which does not contain any marking, nodes or other curves in C) along with a homeomorphism $\varphi: U \rightarrow S^1 \times (-1, 1)$ that restricts to an isometry,

$$U \setminus c \xrightarrow{\sim} S^1 \times (-1, 1) \setminus S^1 \times \{0\}$$

when the left hand side is equipped with the hyperbolic metric from $f(U \setminus c)$ in X and the right hand side has metric $t^2 d\sigma^2 + dt^2/t^2$ where $\sigma \in S^1$ and $t \in (-1, 1) \setminus \{0\}$. We say that $(C, X, f) \sim (D, Y, g)$ if there is an isometry $\psi: X \rightarrow Y$ along with a $\gamma \in \mathbf{Diffeo}^0(S)$ such that $\gamma(C) = D$ and which preserves the markings pointwise. Moreover the diagram,

$$\begin{array}{ccc} S \setminus (C \cup P) & \xrightarrow{\gamma} & S \setminus (D \cup P) \\ f \downarrow & & \downarrow g \\ X & \xrightarrow{\psi} & Y \end{array}$$

commutes and such that γ respects the singular hyperbolic structure in the following sense: For a witness φ of the singular hyperbolic structure on $[C, X, f]$ the morphism $\varphi \circ \gamma^{-1}: \gamma(U) \rightarrow S^1 \times (-1, 1)$ gives a singular hyperbolic structure on (D, Y, g) . $\diamond$

As a set $\overline{\mathcal{T}}_S$ consists of the singular hyperbolic structures on S .

Remark 3.3.7. In particular when $C = \emptyset$ a singular hyperbolic structure (C, X, f) is just a marking of the curve (S, P) . Whenever $C \neq \emptyset$, requiring singular hyperbolic structure is equivalent to saying that the triple (C, X, f) induces a marking of the curve $(S/C, P)$, as essentially we require the smooth structure to be compatible with nodal contractions around the curves c in C . So in essence the singular hyperbolic structure, is just a coherent packaging for the set of markings on any specialization of (S, P) . $\diamond$

We have a natural stratification on the Harvey bordification given by the map,

$$s_S: \overline{\mathcal{T}}_S \rightarrow \mathfrak{C}_S, \quad [C, X, f] \mapsto [C].$$

It coincides with the natural stratification of $\overline{\mathcal{T}}_S$ considered as a manifold with corners [Jan23, p. 21]. Where $\overline{\mathcal{T}}_S$ is a smooth manifold with corners by endowing a smooth structure on the star open $(\overline{\mathcal{T}}_S)_{\geq \sigma}$ which is denoted by $U(\sigma)$ for every $\sigma \in \mathfrak{C}_S$. Namely for any maximal admissible system of curves μ containing σ we have a diffeomorphism

$$U(\sigma) \xrightarrow{\sim} \mathbb{R}_{\geq 0}^\sigma \times \mathbb{R}_{> 0}^{\mu \setminus \sigma} \times \mathbb{R}^\mu, \quad [C, X, f] \mapsto (r_1, \dots, r_{\#\sigma}, \ell_1, \dots, \ell_{\#\mu \setminus \sigma}, \tau_1, \dots, \tau_{\#\mu})$$

where ℓ_i is once again the “length of the unique geodesic isotopic to $f(c_i)$ ” for $c_i \in \mu \setminus \sigma$, the τ_i are the twist parameters of the $c_i \in \mu$ and the r_j is the length of the unique geodesic isotopic to c_j (which can also go to zero) for every $c_j \in \sigma$. The stratification on the product space is defined by

$$\mathbb{R}_{\geq 0}^\sigma \times \mathbb{R}_{> 0}^{\mu \setminus \sigma} \times \mathbb{R}^\mu \xrightarrow{\text{pr}_1} \mathbb{R}_{\geq 0}^\sigma \rightarrow (\mathfrak{C}_S)_{\geq \sigma}, \quad (r_c)_\sigma \mapsto \{c | r_c = 0\}.$$

We define an action $\Gamma_S \curvearrowright \overline{\mathcal{T}}_S$ by,

$$\bar{\gamma} \cdot [C, X, f] = [\gamma(C), X, f \circ \gamma^{-1}]$$

which is compatible with the action $\Gamma_S \curvearrowright \mathfrak{C}_S$ via s_S since $\gamma \cdot [C] = [\gamma(C)]$. Thus, this defines an action on $\overline{\mathcal{T}}_S$ as stratified space. Again one can show that this action is properly discontinuous [Jan23, Theorem 4.5(1)], so that the quotient stack is Deligne Mumford. Moreover, since the quotient space $\Gamma_S \backslash \overline{\mathcal{T}}_S$ is compact by [Jan23, Theorem 4.5(2)] and since we have an isomorphism $[\overline{\mathcal{T}}_S / \Gamma_S] \simeq [\Gamma_S \backslash \overline{\mathcal{T}}_S]$ (in [Jan23, Remark 4.3] it is established that the quotient stack $[G \backslash X]$ is the stack of principal G -bundles together with G -equivariant maps into X , i.e. $[G/X]$). So this stack is compact. In fact when we take (S, P) to be a non-singular stable surface it identifies with the real oriented blowup of $\overline{\mathcal{M}}_{g,n}$

Theorem 3.3.8. [Jan23, Theorem 4.11] Let (S, P) be a non-singular stable surface of genus g . There is an isomorphism,

$$\text{Bl}_\partial(\overline{\mathcal{M}}_{g,P}^{\text{an}}) \xrightarrow{\sim} [\overline{\mathcal{T}}_S / \Gamma_S]$$

As stacks over the category of smooth manifolds with corners.

3.3.2 Interpreting Jansen’s Calculation for Representations

In the computation of the exit path category of the moduli of stable curves, Jansen essentially details how to construct the exodromy representation on $\overline{\mathcal{M}}_{g,n}$ from the $\Gamma_{g,n}$ -equivariant representation. The real oriented blowup of $\overline{\mathcal{M}}_{g,n}$ is isomorphic to $[\overline{\mathcal{T}}_{g,n} / \Gamma_{g,n}]$ on which the exodromy representations of sheaves are much easier to compute by $\Gamma_{g,n}$ -equivariant descent, as we will see in the next chapter. Before writing the exit-path category we define the twisted arrow category.

Definition 3.3.9 (Twisted Arrow Category, [Jan23, Section 5.2]). Let $\mathcal{C}$ be a category. The **twisted arrow category** of $\mathcal{C}$, written $\text{Tw}(\mathcal{C})$, is the category:

(Objects): An object is a morphisms $x \rightarrow y$ in $\mathcal{C}$.

(Morphisms): A morphism $(a, b): (f: x \rightarrow y) \rightarrow (f': x' \rightarrow y')$ is a pair of maps $a: x' \rightarrow x$ and $b: y' \rightarrow y$ such that $f' = b \circ f \circ a$.

(Composition): Composition is component-wise composition in $\mathcal{C}$. ◇

In fact the exit path category can be expressed as,

$$\Pi_1^{\text{Strat}}(\overline{\mathcal{M}}_{g,n}) \cong \text{colim}_{(\sigma \leq \tau) \in \text{Tw}(\text{CL}_S)^{\text{op}}} \Pi_1^{\text{Strat}}(\overline{X}_\tau^{S/\sigma})$$

Indexed over exit path categories of subsets of the universal cover of this quotient stack. Indeed for every $(\sigma \leq \tau) \in \text{Tw}(\text{CL}_S)$ we have a locally closed embedding

$$f_{\sigma,\tau}: \overline{X}_\tau^{S/\sigma} \xrightarrow{\text{open}} \overline{\mathcal{T}}_{S/\sigma} \xrightarrow{\text{closed}} \overline{\mathcal{T}}_{g,n}$$

Where S is the smooth n -marked surface of genus g . Using these maps we can construct our representation on $\overline{\mathcal{M}}_{g,n}$ out of the $\Gamma_{g,n}$ -equivariant representation on $\overline{\mathcal{T}}_{g,n}$ by the universal property of the colimit.

Lemma 3.3.10. A $\Gamma_{g,n}$ -equivariant exodromy representation induces a unique exodromy representation on $\overline{\mathcal{M}}_{g,n}$.

Proof. Let (F, α) be a $\Gamma_{g,n}$ -equivariant representation on $\overline{\mathcal{T}}_{g,n}$. We claim it corresponds to a unique exodromy representation on $\overline{\mathcal{M}}_{g,n}^{\text{Top}}$. For all objects $(\sigma \leq \tau)$ in the twisted arrow category $\text{Tw}(\text{CL}_S)$ define the functor $F_{\sigma,\tau}$ as the composite,

$$F_{\sigma,\tau}: (\mathcal{C}_S)_{/\sigma} \xrightarrow{(f_{\sigma,\tau})_*} \Pi_1^{\text{Strat}}(\overline{\mathcal{T}}_{g,n}) \xrightarrow{F} \mathbf{Set}.$$

Recall that a morphism in the twisted arrow category

$$\gamma: (\sigma \leq \tau) \xrightarrow{\gamma} (\sigma' \leq \tau')$$

corresponds to a mapping class $\gamma \in \Gamma_S$ such that the inequalities $\gamma \cdot \sigma' \leq \sigma$ and $\gamma \cdot \tau' \geq \tau$ hold. The first inequality $\gamma \cdot \sigma' \leq \sigma$, induces the morphism of partially ordered sets

$$(\mathcal{C}_S)_{/\sigma'} \xrightarrow{\gamma \cdot} (\mathcal{C}_S)_{/\sigma}, \quad \alpha \mapsto \gamma \cdot \alpha.$$

We claim that these data assemble into a 2-cocone. Then invoking the universal property of the colimit

$$\Pi_1^{\text{Strat}}(\overline{\mathcal{M}}_{g,n}) \cong \text{colim}_{(\sigma \leq \tau) \in \text{Tw}(\text{CL}_S)^{\text{op}}} \Pi_1^{\text{Strat}}(\overline{X}_\tau^{S/\sigma})$$

we obtain an exodromy representation on $\overline{\mathcal{M}}_{g,n}^{\text{Top}}$.

So we would like to show that the triangles,

$$\begin{array}{ccc} (\mathcal{C}_S)_{/\sigma'} & \xrightarrow{\gamma \cdot} & (\mathcal{C}_S)_{/\sigma} \\ & \searrow F_{\sigma',\tau'} & \swarrow F_{\sigma,\tau} \\ & \mathbf{Set} & \end{array}$$

commute up to natural isomorphism, in order for the data to form a 2-cocone. To this end, we construct a natural isomorphism

$$F_{\sigma,\tau}(\gamma \cdot \sigma'') \xrightarrow{\sim} F_{\sigma',\tau'}(\sigma'')$$

for every morphism in $\text{Tw}(\mathbf{CL}_S)$. Note that by definition of the pushforward on the exit path category,

$$F \circ (f_{\sigma,\tau})_*(\gamma \cdot \sigma'') = F \circ (\gamma_*)(\sigma'').$$

But the maps $f_{\gamma \cdot \sigma, \gamma \cdot \tau} \circ \gamma$ and $\gamma \circ f_{\sigma,\tau}$ are isomorphic. Under the pushforward this gives us a natural isomorphism $\eta: (f_{\gamma \cdot \sigma, \gamma \cdot \tau} \circ \gamma)_* \xrightarrow{\sim} (\gamma \circ f_{\sigma,\tau})_*$. By whiskering F with η we obtain a natural isomorphism,

$$F \circ (f_{\sigma,\tau} \circ (\gamma_*))(\sigma'') \xrightarrow{\sim} F \circ (\gamma_*) \circ (f_{\sigma,\tau})_*(\sigma'').$$

Additionally, consider the map

$$i_\gamma: \overline{\mathcal{T}}_{g,n} \hookrightarrow \Gamma_S \times \overline{\mathcal{T}}_{g,n}, \quad x \mapsto (\gamma, x).$$

Recalling that the symmetry groupoid of the stack is $(s, t): \Gamma_{g,n} \times \overline{\mathcal{T}}_{g,n} \rightrightarrows \overline{\mathcal{T}}_{g,n}$ we see that $s \circ i_\gamma$ is $\text{id}_{\overline{\mathcal{T}}_{g,n}}$ while $t \circ i_\gamma = \gamma(-)$. So whiskering the natural transformation α witnessing the Γ_S -equivariance of F with the functor $(i_\gamma)_* \circ (f_{g,n})_*$ we obtain a natural isomorphism,

$$\alpha * (i_\gamma)_*: F_{\sigma,\tau} \xrightarrow{\sim} F \circ (\gamma)_* \circ (f_{\sigma,\tau})_*.$$

Denote the composite with $F * \eta$,

$$F_{\sigma,\tau} \xrightarrow{\sim} F \circ (\gamma)_* \circ (f_{\sigma,\tau})_* \xrightarrow{\sim} F_{g,n} \circ (\gamma)_*$$

by α_γ . This α_γ witnesses the commutativity of our triangle. Consequently, by the universal property of the 2-colimit we obtain a representation,

$$\bar{F}: \Pi_1^{\text{Strat}}(\overline{\mathcal{M}}_{g,n}) \rightarrow \mathbf{Set}.$$

Conversely, we precompose an exodromy representation G on $\Pi_1^{\text{Strat}}(\overline{\mathcal{M}}_{g,n})$ with the pushforward on exit path categories of the projection $\pi: [\overline{\mathcal{T}}_{g,n}/\Gamma_{g,n}] \rightarrow \overline{\mathcal{M}}_{g,n}^{\text{Top}}$, to obtain a representation $G \circ \pi_*$. Then $\Gamma_{g,n}$ -equivariant descent for representations gives us a representation (F', α') . $\square$

In the next chapter we compute the $\Gamma_{g,n}$ -equivariant representation of a sheaf and its associated representation on $\overline{\mathcal{M}}_{g,n}$ for certain values of g, n .

Chapter 4

Describing the Exodromy Representation of $\mathbf{R}^1\pi_*\underline{\mathbb{Q}}$ on $\overline{\mathcal{M}}_{g,n}^{\text{Top}}$

“Because, on reflection, once you have seen your own face and recognized the colour of your eyes, tasted the air and smelled the soil, drunk from the purest fountains and the dirtiest wells, that is the kind of thing you can say about life. It’s not nothing.”

— *The Seven Moons of Maali Almeida* by Shehan Karunatilaka

Let $\pi: \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ be the universal curve. Our primary objective in this chapter will be to describe the exodromy representation of the sheaf $\mathbf{R}^1\pi_*\underline{\mathbb{Q}}$ in as much detail as possible. Jansen’s inductive strategy for computing the exit path category informs a practical approach for computing exodromy representations on this moduli stack [Jan23, Section 6]. Lemma 3.3.10 which is a direct reformulation of this consequence of Jansen’s result, expresses the exodromy representation of a constructible sheaf on the moduli of stable curves as a colimit of said exodromy representation pulled back to the real oriented blowup. As we will see shortly, computing the exodromy representation of our sheaf on this space is greatly simplified by the isomorphism [Jan23, Appendix B],

$$\text{Bl}_{\partial}(\overline{\mathcal{M}}_{g,n}^{\text{an}}) \xrightarrow{\sim} [\overline{\mathcal{T}}_{g,n}/\Gamma_{g,n}]$$

where $[\overline{\mathcal{T}}_{g,n}/\Gamma_{g,n}]$ is the quotient stack formed by the bordified Teichmüller space and its $\Gamma_{g,n}$ -action $\bar{\gamma} \cdot [C, X, f] := [\gamma(C), X, f \circ \gamma^{-1}]$. In fact in the pullback square,

$$\begin{array}{ccc} [\overline{\mathcal{T}}_{g,n}/\Gamma_{g,n}] \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{C}}_{g,n} & \longrightarrow & \overline{\mathcal{C}}_{g,n} \\ \pi' \downarrow & \lrcorner & \downarrow \pi \\ [\overline{\mathcal{T}}_{g,n}/\Gamma_{g,n}] & \longrightarrow & \overline{\mathcal{M}}_{g,n} \end{array}$$

the map π is proper. Although this is a well-known fact, we spell this out here. This follows by [ACG11, The proof of lemma XII.10.1], in which it is established that the pullback

$$\overline{\mathcal{C}}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}} S_{g,n}$$

corresponds to a family $X \rightarrow S_{g,n}$ of stable n -marked curves of genus g . This family is of finite type and proper by definition and hence so is the underlying analytical family by [Ray71, Proposition 3.1, Proposition 3.2]. So by the proper base change theorem, we simply compute the exodromy representation of $\mathbf{R}^1\pi'_*\underline{\mathbb{Q}}$. This will be the content of the first section of this chapter. In the second section we use this result to compute exodromy representations of $\mathbf{R}^1\pi_*\underline{\mathbb{Q}}$ on $\overline{\mathcal{M}}_{g,n}$ for several (g, n) .

4.1 Exodromy of $\mathbf{R}^1\pi'_*\underline{\mathbb{Q}}$ on $[\overline{\mathcal{T}}_{g,n}/\Gamma_{g,n}]$

This computation can be further reduced via equivariant descent. By the descent result lemma 2.3.18 the exodromy representation of the sheaf is equivalent to some $\Gamma_{g,n}$ -equivariant representation (see definition 2.3.17). As we will see this point of view is much more tractable (as this is an equivariant exodromy representation of a sheaf of topological spaces) and also remarkably intuitive (see remark 4.1.3). Since π' is proper, we can once again apply the proper base change theorem:

$$\begin{array}{ccc} \overline{\mathcal{T}}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{C}}_{g,n} & \longrightarrow & [\overline{\mathcal{T}}_{g,n}/\Gamma_{g,n}] \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{C}}_{g,n} \\ \pi'' \downarrow & \lrcorner & \downarrow \pi' \\ \overline{\mathcal{T}}_{g,n} & \longrightarrow & [\overline{\mathcal{T}}_{g,n}/\Gamma_{g,n}] \end{array}$$

to conclude that the pullback of $\mathbf{R}^1\pi'_*\underline{\mathbb{Q}}$ is simply $\mathbf{R}^1\pi''_*\underline{\mathbb{Q}}$. By abuse of notation, we will henceforth denote this sheaf by $\mathbf{R}^1\pi_*\underline{\mathbb{Q}}$.

4.1.1 The Sheaf is Actually Constructible

To apply the exodromy correspondence, we need that the sheaf in question is constructible with respect to the chosen stratification. This follows quite straightforwardly from our reduction. Above each τ -stratum, the universal curve π'' is isomorphic to projection on the first coordinate of the first component in the product $(\{0\}^\tau \times \mathbb{R}_{>0}^{\mu \setminus \tau} \times \mathbb{R}^\mu) \times S/\tau$ under the Fenchel-Nielsen coordinates (since it is a fiber bundle over a contractible space). So $\mathbf{R}^1\pi''_*\underline{\mathbb{Q}}|_{(\overline{\mathcal{T}}_{g,n})_\tau} \cong \mathbf{R}^1\mathrm{pr}_{1,*}\underline{\mathbb{Q}}$. But this sheaf is constant, so $\mathbf{R}^1\pi''_*\underline{\mathbb{Q}}$ is constructible. So by $\Gamma_{g,n}$ -equivariant descent, so is $\mathbf{R}^1\pi'_*\underline{\mathbb{Q}}$. So by [Jan23, Theorem 6.4], so is $\mathbf{R}^1\pi_*\underline{\mathbb{Q}}$.

4.1.2 A Conjecture for The Case $(g, n) = (1, 1)$

In this section, we explain the approach we believe one would need to take to compute the $\Gamma_{g,n}$ -equivariant exodromy representation of the sheaf $\mathbf{R}^1\pi'_*\mathbb{Q}$ in the case $(g, n) = (1, 1)$. Due to time constraints, we did not determine the geometry of the universal curve pulled back to the bordified Teichmüller space. Although we have a pretty good idea of what this could be. We claim the pullback

$$\pi'' : \overline{\mathcal{T}}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}^{\text{an}}} \overline{\mathcal{C}}_{g,n}^{\text{an}}$$

is a family of nodal Riemann surfaces. Taking inspiration from [ACG11, Lemma X.9.19] we make a conjecture about the geometry of the space. But first we need to define what a transverse family is.

Definition 4.1.1 (Transverse Family). Let S be an analytic space, and consider a family of nodal Riemann surfaces $f : X \rightarrow S$ over S . For any $s \in S$ let $p_1, \dots, p_k$ be the nodes of the fiber $X_s = f^{-1}(s)$. Since each p_i is a nodal singularity, locally in X around each p_i , the family f is of the form $xy = t_i$ where $t_i \in D(0; \epsilon)$. We say that f is **transverse at** s if the coordinates $t_1, \dots, t_k$ form a system of local coordinates on some neighborhood of s . We say that f is transverse if it is transverse for every $s \in S$. $\diamond$

Conjecture 4.1.2. Suppose the family $f : \mathfrak{X} := \overline{\mathcal{T}}_{1,1} \times_{\overline{\mathcal{M}}_{1,1}^{\text{an}}} \overline{\mathcal{C}}_{1,1}^{\text{an}} \rightarrow \overline{\mathcal{T}}_{1,1}$ is transverse. Let $b_0 \in \overline{\mathcal{T}}_{1,1}$ be in the $\sigma = \{\sigma_1, \dots, \sigma_k\}$ -stratum, and consider the fiber $C_0 = f^{-1}(b_0)$ with nodes $\{p_1, \dots, p_k\}$. Then there exists an open neighborhood $U \ni b_0$ and a pair of retractions $r : U \rightarrow \{b_0\}$ and $R : \mathfrak{X}|_U \rightarrow C_0$ (with R living over r), which are compatible with the markings $m_1, \dots, m_n$. Moreover, given a smooth fiber C in $\mathfrak{X}|_U$, the composite

$$p : C \hookrightarrow \mathfrak{X}|_U \xrightarrow{R} C_0$$

has the property that each $p^{-1}(p_i)$ is isotopic to σ_i on C . $\diamond$

By lemma 4.1.5 of the next section, we then deduce that the associated cospecialization map,

$$f : H^1(C_x, \mathbb{Q}) \xrightarrow{\sim} \mathcal{F}_x \rightarrow \mathcal{F}_y \xrightarrow{\sim} H^1(C_y, \mathbb{Q})$$

induced by the exit path $\sigma \leq \emptyset$ in $\overline{\mathcal{T}}_{1,1}$ is canonically isomorphic to $p^* : H^1(S, \mathbb{Q}) \rightarrow H^1(S/\sigma, \mathbb{Q})$. As we will see later on, under our conjecture, the $\Gamma_{1,1}$ -equivariant data is directly inferred from the action

$$\Gamma_{1,1} \times \overline{\mathcal{T}}_{g,n} \rightrightarrows \overline{\mathcal{T}}_{g,n}$$

to be $\alpha_{\overline{\gamma},x} = (\gamma^{-1})^*$ on the smooth stratum. The equivariance on the singular stratum is follows from the equivariance on the smooth stratum by naturality of α . Since the mapping classes preserve the strata, the parallel transport maps are trivial in the sense that they are given by,

$$(\mathbf{R}^1\pi'_*\mathbb{Q})_{\gamma \cdot x} \rightarrow (\mathbf{R}^1\pi'_*\mathbb{Q})_x, [s]_{\gamma \cdot x} \mapsto [s]_x.$$

We recover the monodromy of the space from the equivariance data followed by the parallel transport since the space itself is exit-path simply connected in the sense that its stratified fundamental group is its stratifying poset, this gives us the next heuristic for interpreting the $\Gamma_{g,n}$ -equivariance data.

Remark 4.1.3 (Equivariant Representation = Cospecialization + Monodromy). Let $G \curvearrowright X$ and (X, P) be a conically stratified space such that $\Pi_1^{\text{Strat}}(X) = P$. Let $(\rho_{\mathcal{F}}, \alpha)$ be a G -equivariant exodromy representation on X . The first term, $\rho_{\mathcal{F}}$ encodes the cospecialization data along the strata. We assume the cospecializations $\rho_{\mathcal{F}}(\gamma)$ are all injective. Then the equivariance term α encodes the monodromy induced by the group action. This follows from naturality of $\rho(\alpha)$: $\rho_{\mathcal{F}} \circ s^* \Rightarrow \rho_{\mathcal{F}} \circ t^*$ since the cospecialization maps are injective so the values on singular strata can be read off of diagram,

$$\begin{array}{ccc} \rho(q) & \xrightarrow{F \circ s^*(\gamma)} & \rho(\text{smooth}) \\ \rho(\alpha)(\bar{\gamma}, \sigma) \downarrow & & \downarrow \rho(\alpha)(\bar{\gamma}, \text{smooth}) \\ \rho(q) & \xrightarrow{F \circ t^*(\gamma)} & \rho(\text{smooth}). \end{array} \quad \diamond$$

4.1.3 Determining the Cospecialization Data

In this section, we describe the cospecialization map $f: \mathcal{F}_x \rightarrow \mathcal{F}_y$ associated to the homotopy class of some exit path $\gamma: x \rightsquigarrow y$. This should become less tedious using the following two lemmata.

Lemma 4.1.4. Let $x \in \overline{\mathcal{T}}_{g,n}$ be a point. The open neighborhoods $V \ni x$ deformation retracting on x are coinital among all neighborhoods of x . So there is an isomorphism $\mathcal{F}_x \cong \varinjlim_{V \ni x \text{ contractible}} \mathcal{F}(V)$.

Proof. The manifold with corners $\overline{\mathcal{T}}_{g,n}$ is locally contractible, so indeed the contractible neighborhoods form a fundamental system of neighborhoods and are therefore coinital. $\square$

The next lemma shows that all exit paths on $\overline{\mathcal{T}}_{g,n}$ are pure with respect to the sheaf in question.

Lemma 4.1.5 (The Cospecialization Maps). Consider the universal curve

$$\pi': \overline{\mathcal{C}}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{T}}_{g,n} \rightarrow \overline{\mathcal{T}}_{g,n}.$$

The sheaf $\mathbf{R}^1\pi'_*\underline{\mathbb{Q}}$ is constructible and the diagram

$$\begin{array}{ccccccc} H^1(\pi^{-1}(x), \mathbb{Q}) & \xleftarrow{i_x^*} & H^1(\pi^{-1}(W), \mathbb{Q}) & \xrightarrow{i^*} & H^1(\pi^{-1}(V), \mathbb{Q}) & \xrightarrow{i_y^*} & H^1(\pi^{-1}(y), \mathbb{Q}) \\ & \searrow \cong & \downarrow \lambda_W & & \downarrow \lambda_V & \swarrow \cong & \\ & & (\mathbf{R}^1\pi'_*\underline{\mathbb{Q}})_x & \xrightarrow{\text{cosp}_\gamma} & (\mathbf{R}^1\pi'_*\underline{\mathbb{Q}})_y & & \end{array}$$

commutes. Suppose moreover that $\pi^{-1}(W)$ (and $\pi^{-1}(V)$) deformation retract onto $\pi^{-1}(x)$ (resp. $\pi^{-1}(y)$) and label the retraction by r_x^W (resp. r_y^V). Then i_x^*, i_y^* are isomorphisms and the cospecialization is given by retract $\left(r_x^W \Big|_{\pi^{-1}(y)} \right)^*$.

Proof. We have already established that the sheaf is constructible. We show that the diagram commutes. The stalk of a sheaf can be computed at the presheaf level so

$$(\mathbf{R}^1\pi'_*\underline{\mathbb{Q}})_x \xrightarrow{\sim} \varinjlim_{W \ni x} H^1(\pi^{-1}(W), \underline{\mathbb{Q}})$$

has the property that each $p^{-1}(p_i)$ is isotopic to σ_i on C . $\diamond$

Conjecture 4.1.7 (The cospecialization map). Let $\gamma: x \rightsquigarrow y$ be the exit path associated to $\sigma_1 \leq \sigma_2$. Then the cospecialization map expressed on cohomology as

$$H^1(S/\sigma_1, \mathbb{Q}) \xrightarrow{\sim} (\mathbb{R}^1\pi_*\mathbb{Q})_x \rightarrow (\mathbb{R}^1\pi_*\mathbb{Q})_y \xrightarrow{\sim} H^1(S/\sigma_2, \mathbb{Q})$$

is $(p_{\sigma_1}^{\sigma_2})^*$ where,

$$p_{\sigma_1}^{\sigma_2}: S/\sigma_2 \rightarrow S/\sigma_1$$

is the projection map.

Proof. Let x be in the σ_1 -stratum of $\overline{\mathcal{F}}_{g,n}$. By the previous conjecture, there exists a contractible open neighborhood $V \ni x$ together with an explicit retract $r_{\sigma_1}: \pi^{-1}(V) \rightarrow \pi^{-1}(x)$. Similarly, there exists such a contractible open U of y in the σ_2 -stratum. We then have a composite,

$$H^1(\pi^{-1}(x), \mathbb{Q}) \xrightarrow[(r_{\sigma_1})^*]{(i_x)^*} H^1(\pi^{-1}(V), \mathbb{Q}) \xrightarrow{i^*} H^1(\pi^{-1}(U), \mathbb{Q}) \xrightarrow{i_y^*} H^1(\pi^{-1}(y), \mathbb{Q})$$

which is the cospecialization map by lemma 4.1.5. By the explicit description of r_{σ_1} given in the previous conjecture, this map is given by, $f \mapsto p_{\sigma_1}^{\sigma_2}(f)$ on topological spaces, thus we indeed get $(p_{\sigma_1}^{\sigma_2})^*$ as our cospecialization. $\square$

In fact, if this were to be the case, we can directly piece this result together globally.

Conjecture 4.1.8 (Global cospecializations). Let $x_\sigma \rightsquigarrow x_\tau$ be an exit path on $\overline{\mathcal{F}}_{g,n}$ between a point x_σ in the σ -stratum and x_τ in the τ -stratum. The cospecialization of this map on the sheaf $\mathbf{R}^1\pi_*\underline{\mathbb{Q}}$ on $\overline{\mathcal{F}}_{g,n}$ is the pushforward of the nodal degeneration,

$$H^1(S/\sigma; \mathbb{Q}) \rightarrow H^1(S/\tau; \mathbb{Q})$$

whose cokernel consists of the vanishing cycles associated to the elements $\sigma \setminus \tau$. Giving us the exodromy representation in full:

$$\begin{aligned} F: \Pi_1^{\text{Strat}}(\overline{\mathcal{F}}_{g,n}) &\rightarrow \mathbf{Set}, \\ x_\sigma &\mapsto H^1(S/\sigma; \mathbb{Q}) \\ (x_\sigma \leq x_\tau) &\mapsto H^1(S/\sigma; \mathbb{Q}) \xrightarrow{(p_\sigma^\tau)^*} H^1(S/\tau; \mathbb{Q}). \end{aligned}$$

Proof. Given any $\sigma_1 \leq \sigma_2 \leq \sigma_3$ we have that,

$$p_{\sigma_1}^{\sigma_2} \circ p_{\sigma_2}^{\sigma_3} = p_{\sigma_1}^{\sigma_3}$$

so cospecialization is functorial. $\square$

4.1.4 The Conjectured $\Gamma_{g,n}$ -Equivariance Data

It remains for us to conjecture the $\Gamma_{g,n}$ -equivariance data of the representation. In this section, we establish that this is equivalent to determining the representation

$$\Psi: \Gamma_{g,n} \rightarrow \mathrm{SL}(2g + n - 1, \mathbb{Z})$$

associated to the action on cohomology $\Gamma_{g,n} \curvearrowright H^1(S_{g,n})$ (see [FM11, Section 6.3] for the argument that $\Gamma_g \curvearrowright H_1$ lands in $\mathrm{SL}(2g, \mathbb{Z})$). Fortunately the equivariance data can be computed using the tools setup in the previous section. Under the equivalence established in theorem 2.3.12, the equivariance data,

$$\alpha: s^*\mathbf{R}^1\pi_*\underline{\mathbb{Q}} \rightarrow t^*\mathbf{R}^1\pi_*\underline{\mathbb{Q}}$$

is given by $\alpha(U) := \mathbf{R}^1\pi_*\underline{\mathbb{Q}}(\phi|_U)$ where $\phi: s^*x \rightarrow t^*x$ is the tautological morphism induced by id_Γ , the identity on the symmetry groupoid. In our case the symmetry groupoid is the transformation groupoid

$$\Gamma_{g,n} \times \overline{\mathcal{F}}_{g,n} \rightrightarrows \overline{\mathcal{F}}_{g,n}.$$

Keeping this in mind we see that at an element $(\overline{\gamma}, x) \in \Gamma_{g,n} \times \overline{\mathcal{F}}_{g,n}$ the equivariance data is given by

$$\alpha_{(\overline{\gamma}, x)} = \mathbf{R}^1\pi_*\underline{\mathbb{Q}}(\phi_{(\overline{\gamma}, x)}) = \mathbf{R}^1\pi_*\underline{\mathbb{Q}}((\overline{\gamma}, x)).$$

Since the symmetry groupoid is isomorphic to the groupoid $\mathrm{Isom}(x_{\mathrm{versal}}, x_{\mathrm{versal}})$ where x_{versal} is the versal family, the collection of morphisms $\left(y \xrightarrow{(\overline{\gamma}, y)} \gamma \cdot y\right)_{y \in \overline{\mathcal{F}}_{g,n}}$ define a global automorphism of the versal family which we label γ . Note how $\mathbf{R}^1\pi_*\underline{\mathbb{Q}}(\gamma)$ is an endomorphism of $x_{\mathrm{versal}}^*\mathbf{R}^1\pi_*\underline{\mathbb{Q}}$ which is our sheaf $\mathbf{R}^1\pi_*''\underline{\mathbb{Q}}$.

Conjecture 4.1.9 (Monodromy on the Smooth Locus). Let $x \in \overline{\mathcal{F}}_{g,n}$ be the chosen basepoint. The map $\mathbf{R}^1\pi_*\underline{\mathbb{Q}}((\overline{\gamma}, x))$ is the parallel transport of $(\overline{\gamma}, x)$ on $\mathbf{R}^1\pi_*''\underline{\mathbb{Q}}$. This is once again governed by the Picard-Lefschetz transformation of the family,

$$\overline{\mathcal{F}}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}^{\mathrm{an}}} \overline{\mathcal{C}}_{g,n}^{\mathrm{an}} \rightarrow \overline{\mathcal{F}}_{g,n}.$$

Let $x_\emptyset$ be a point in the open stratum. Then,

$$\rho(\alpha)(\overline{\gamma}, x_\emptyset) = (\mathrm{PL}(\gamma^{-1}))^* = \Psi(\gamma). \quad \diamond$$

Under the exodromy correspondence, we obtain a natural isomorphism,

$$\rho(\alpha): F \circ s_* \xrightarrow{\sim} F \circ t_*.$$

We can describe this natural transformation explicitly now. Recall the representation

$$\Psi: \Gamma_{g,n} \rightarrow \mathrm{SL}(2g + n - 1, \mathbb{Z})$$

associated to the action on cohomology $\Gamma_{g,n} \curvearrowright H^1(S_{g,n})$. The next theorem describes $\rho(\alpha)$ explicitly in terms of Ψ , showing that the exodromy representation of $\mathbf{R}^1\pi'_*\underline{\mathbb{Q}}$ is governed by Ψ .

Conjecture 4.1.10. The exit path category $\Pi_1^{\text{Strat}}(\Gamma_{g,n} \times \overline{\mathcal{T}}_{g,n}^{\text{Top}})$ is the poset $\Gamma_{g,n} \times \mathbb{C}_{S_{g,n}}$. When we express $\rho(\alpha)$ in terms of the cohomology at the points x_σ in the σ -stratum we get the explicit description

1. for pair $(\overline{\gamma}, x_\emptyset)$, we have that $\rho(\alpha)(\overline{\gamma}, x) = \Psi(\overline{\gamma})$.
2. for a general element $(\overline{\gamma}, x_\sigma)$ we have that $\rho(\alpha)(\overline{\gamma}, x_\sigma)$ is the restriction of $\Psi(\overline{\gamma})$ to $H^1(S/\sigma)$ under $(p_\sigma^\emptyset)^*$.

So $\rho(\alpha)$ is determined by Ψ and hence, given our explicit description of the cospecialization maps.

Proof. The case on the smooth stratum follows directly from conjecture 4.1.9. Now since the cospecialization maps are injective, the general case follows directly from the smooth case by naturality of $\rho(\alpha)$:

$$\begin{array}{ccc} H^1(S_{g,n}/\sigma) & \xrightarrow{Fos^*(\gamma, \sigma \leq \emptyset)} & H^1(S_{g,n}) \\ \rho(\alpha)(\overline{\gamma}, x_\sigma) \downarrow & & \downarrow \rho(\alpha)(\overline{\gamma}, x_\emptyset) \\ H^1(S_{g,n}/\sigma) & \xrightarrow{Fot^*(\gamma, \sigma \leq \emptyset)} & H^1(S_{g,n}). \end{array}$$

As a result the natural transformation $\rho(\alpha)$ is determined by Ψ . □

Having reduced the exodromy functor of $\mathbf{R}^1\pi'_*\underline{\mathbb{Q}}$ to the representation Ψ and the explicit cospecializations, we have essentially conjectured the exodromy representation of $\mathbf{R}^1\pi'_*\underline{\mathbb{Q}}$ on $[\overline{\mathcal{T}}_{g,n}/\Gamma_{g,n}]$. Recalling lemma 3.3.10, we can now infer directly what the exodromy representations are from the exit path categories, this is the content of the next section.

4.2 Exodromy of $\mathbf{R}^1\pi_*\underline{\mathbb{Q}}$ on $\overline{\mathcal{M}}_{g,n}$

A morphism in Jansen's Charney-Lee category $\mathbf{CL}_{S_{g,n}}$, the exit path category of $\overline{\mathcal{M}}_{g,n}^{\text{Top}}$, can be canonically factorized as monodromy in the starting stratum followed by a pure exit path: A generic morphism $\sigma \xrightarrow{\gamma} \tau$ is of the form $\gamma \cdot \sigma \leq \tau$. But this can be factorized as the chain $\sigma \xrightarrow{\gamma} \gamma \cdot \sigma \xrightarrow{\text{id}} \tau$. Moreover,

$$\rho_{\mathcal{F}}(\sigma \rightarrow \tau) = \rho_{\mathcal{F}}(\gamma \cdot \sigma \rightarrow \tau) \circ \rho_{\mathcal{F}}(\sigma \rightarrow \gamma \cdot \sigma).$$

By lemma 3.3.10, this evaluates to,

$$\rho_{\mathcal{F}}(\gamma \cdot \sigma \rightarrow \tau) = F_{\gamma \cdot \sigma, \tau}(\gamma \cdot \sigma \leq \tau) \circ \alpha_{\gamma, x_\sigma}.$$

Consequently, the exodromy representation can be understood directly from the representation on the real-oriented blowup $[\overline{\mathcal{T}}_{g,n}^{\text{Top}}/\Gamma_{g,n}]$. So we obtain the following corollary directly from the results of the previous section.

Conjecture 4.2.1 (The Exodromy Representation of $\mathbf{R}^1\pi_*\mathbb{Q}$ on $\overline{\mathcal{M}}_{g,n}$). The Exodromy Representation of $\mathbf{R}^1\pi_*\mathbb{Q}$ on $\overline{\mathcal{M}}_{g,n}$ is given in full (where $(\gamma)^*|_\sigma$ is the restriction of γ^* to $H^1(S/\sigma; \mathbb{Q})$) by:

$$\begin{aligned} \rho_{\mathbf{R}^1\pi_*\mathbb{Q}}: \mathbf{CL}_{S_{g,n}} &\rightarrow \mathbf{Set}, \\ \sigma &\mapsto H^1(S/\sigma; \mathbb{Q}) \\ (\sigma \xrightarrow{\gamma} \tau) &\mapsto \left(H^1(S/\sigma; \mathbb{Q}) \xrightarrow{(\gamma^*)|_\sigma} H^1(S/\gamma \cdot \sigma; \mathbb{Q}) \xrightarrow{(p_{\gamma \cdot \sigma}^\tau)^*} H^1(S/\tau; \mathbb{Q}) \right) \end{aligned} \quad \diamond$$

This is where the journey ends.

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